A TREATISE

ON THE

ADJUSTMENT OF OBSERVATIONS,

WITH

APPLICATIONS TO GEODETIC WORK AND OTHER MEASURES OF PRECISION.

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O Messkunst Zaum der Phantasie,
Wer dir will folgen irret nie
—HALLER.

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PREFACE.

IN the following treatise I have endeavored to give a systematic account of the method of adjusting observations founded on the principle of the mean. The more important applications, especially with reference to geodetic and astronomical work, are fully discussed.

It has been my aim throughout to be practical. The book originated and grew amid actual work, and hence subjects that are interesting mainly because they are curious, and methods of reduction that have become antiquated, are not noticed.

Several of the views enunciated are not in the usual strain, but they are, however, such as I think all experienced observers, though perhaps not all mathematicians, will at once assent to.

As regards notation I have been conservative, usually following Encke's system as given by Chauvenet. Some minor changes have been introduced which it is thought will tend to greater clearness and uniformity of expression.

The examples and illustrations have been drawn chiefly from American sources, for the reason that much valuable material of this kind is to be had, and that thus far it has not been used for this purpose. They have been taken from records of work actually done, and principally from work with which I have been connected.

In the applications to practical work I have aimed at giving only so much of methods of observing as would serve to make the methods of adjustment intelligible. It has been difficult to do this succinctly and at the same time satisfactorily, and accordingly references are given to books where descriptions of instruments and modes of using them can be found.

Special attention has been given to the explanation of checks of computation, of approximate methods of adjustment, and of approximate methods of finding the precision of the adjusted values. But in order to see how far it is allowable to use these short cuts the rigid methods must first be derived. It is for this reason principally that the subject of triangulation has been dwelt on at such length. In general it is unnecessary to spend a great amount of time in finding the probable error, when, after it has been found, it in many cases tells so little.

I have been careful to give references to original authorities as far as I could ascertain them, and also to give lists of memoirs on special subjects which will be of use to any one desiring to follow those subjects farther. Of recent writers I am indebted chiefly to Helmert and Zachariæ. I desire also to acknowledge my obligations to my old Lake Survey friends, Messrs. C. C. Brown, J. H. Darling, E. S. Wheeler, R. S. Woodward, and A. Ziwet, who have read the manuscript in part and given me the benefit of their advice. Mr. Brown deserves special mention for assistance rendered while the book was passing through the press.

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THE ADJUSTMENT OF OBSERVATIONS.

CHAPTER I.

INTRODUCTION.

THE factors that enter into the measurement of a quantity are, the observer, the instrument employed, and the conditions under which the measurement is made.

1. **The Instrument.**—If the measure of a quantity is determined by untrained estimation only, the result is of little value. The many external influences at work hinder the judgment from deciding correctly. For example, if we compare the descriptions of the path of a meteor as given by a number of people who saw the meteor and who try to tell what they saw, it would be found impossible to locate the path satisfactorily. The work of the earlier astronomers was of this vague kind. There was no way of testing assertions, and theories were consequently plentiful.

The first great advance in the science of observation was in the introduction of instruments to aid the senses. The instrument confined the attention of the observer to the point at issue and helped the judgment in arriving at conclusions. As with a rude instrument different observers would get the same result, it is not to be wondered at that for a long time a single instrumental determination was considered sufficient to give the value of the quantity measured.

The next advance was in the questioning of the instrument and in showing that a result better on the whole than a single direct measurement could be found. This opened

the way for better instruments and better methods of observation. For example, Gascoigne's introduction of cross-hairs into the focus of the telescope led to better graduated circles and to better methods of reading them, resulting finally in the reading microscopes now almost universally used. The culminating point was reached by Bessel, who, by his systematic and thorough investigation of instrumental corrections and methods of observation, may be said to have almost exhausted the subject. He confined himself, it is true, to astronomical and geodetic instruments, but his methods are of universal application.

The questioning of an instrument naturally arises from noticing that there are discrepancies in repeated measurements of a magnitude with the same instrument or in measures made with different instruments. Thus, if a distance was measured with an ordinary chain, and then measured with a standard whose length had been very carefully determined, and the two measurements differed widely, we should suspect the chain to be in error and proceed to examine it before further measuring. So discrepancies found in measurements made with the same measure at different temperatures have shown the necessity of finding the length of the measure at some fixed temperature, and then applying a correction for the length at the temperature at which the measurement is made.

Corrections to directly measured values are thus seen to be necessary, and to be due to both internal and external causes. The internal causes arising from the construction of the instrument are seen to be in great measure capable of elimination. From geometrical considerations the observer can tell the arrangement of parts demanded by a perfect instrument. He can compute the errors that would be introduced by certain supposed irregularities in form and changes of condition. The instrument-maker cannot, it is true, fulfil the conditions necessary for a perfect instrument, but he has been gradually approaching them more and more closely. It is to be remembered that, even if an

instrument could be made perfect at any instant, it would not remain so for any great length of time.

It hence followed as the next great advance that the instrument was made adjustable in most of its parts, so that the relative positions of the parts are under the control of the observer. This is getting to be more and more the case with the better class of instruments.

Not only is error diminished by the improved construction of the instrument, but also by more refined methods of handling it. It may be, indeed, that some contrivances beyond those required to make necessary readings for the measure of the quantity in question may be needed. Thus, with a graduated circle regular or periodic errors of graduation may be expected. If the angle between two signals were read with a theodolite, the reading on each signal, and consequent value of the angle, would be influenced by the periodic errors of the circle of the instrument. Though a single vernier or microscope would suffice to read the circle when the telescope is directed to the signals, yet, as the circle is incapable of adjustment, we can only get rid of the influence of the periodicity by employing a number of verniers or microscopes placed at equal intervals around the circle. It happens that this same addition of microscopes eliminates eccentricity of the graduated circle as well.

This same principle of making the method of observation eliminate the instrumental errors is carried through even after the nicest adjustments have been made. Thus, in ordinary levelling, if the backsights and foresights are taken exactly equal the instrumental adjustment may be poor and still good work may be done. But good work is more likely if the adjustments have been carefully made, as if for unequal sights, and still the sights are taken equal.

Simplicity of construction in an instrument is also to be aimed at. An instrument that theoretically *ought* to work perfectly is often a great disappointment in practice. Two striking examples are the compensating base-apparatus and the repeating theodolite, both of which have been aban-

doned on all the leading surveys. In both cases the mechanical difficulties in the way have proved insurmountable, and the instruments have been replaced by others of simpler construction, to whose readings corrections can either be computed and applied or the errors of the readings can be eliminated by the method of observation. In this way no hopes of an accuracy which cannot be realized are held out.

Such is the perfection now attained in the construction of mathematical instruments, and the skill with which they can be manipulated, that comparatively little trouble in making observations arises from the instrument itself.

2. **External Conditions.**—The great obstacles to accurate work arise from the influence of external conditions—conditions wholly beyond the observer's or instrument-maker's control, and whose effect can, in general, neither be satisfactorily computed nor certainly eliminated by the method of observation. We have no means of finding the complex laws of their action. Many of them can be *avoided* by not observing while they operate in any marked degree. Thus, if while an observer was reading horizontal angles on a lofty station a strong wind should spring up, it would be useless for him to continue the work. If the air commenced to "boil" he should stop. If the sun shone on one side of his instrument its adjustments would be so disturbed that good work could not be expected. So in comparisons of standards. Comparisons made in a room subject to the temperature variations of the outside air would be of little value. The standards should not only not be exposed to sudden temperature changes during comparisons, but at no other time; for it has been shown by recent experiments* that the same standard may have different lengths at the same temperature after exposure to wide ranges of temperature.*

The effects of external disturbances may sometimes be eliminated, in part at least, by the method of observation.

* *American Journal of Science*, July, 1881.

In the measurement of horizontal angles where the instrument is placed on a lofty station, the influence of the sun causes the centre post or tripod of the station to twist in one direction during the day. When this influence is removed at night the twist is in the opposite direction. Assuming the twist to act uniformly, its effect on the results is eliminated by taking the mean of the readings on the signals observed in order of azimuth and then immediately in the reverse order.

Atmospheric refraction is another case in point. In observing for time with an astronomical transit the effect of refraction on the mean of the recorded readings is eliminated by taking the star on the same number of threads on each side of the middle thread. On the other hand, in the measurement of horizontal angles, if long lines are sighted over, or lines passing from land over large bodies of water or over a country much broken, the effects of refraction are apt to be very marked. As we have no means of eliminating the discordances arising in this way by the method of observation, all we can do is, while planning a triangulation, to avoid as far as possible the introduction of such lines.

It may happen that the effect of the external disturbances on the observations can be computed approximately from theoretical considerations assuming a certain law of operation. If the correction itself is small this is allowable. As an example take the zenith telescope, with which the method of observing for latitude is such that the correction for refraction is so small that the error of the computed value is not likely to exceed other errors which enter into the work.

3. **The Observer.**—Lastly we come to the observer himself as the third element in making an observation. Like the external conditions, he is a variable factor; all new observers certainly are.

The observer, having put his instrument in adjustment and satisfied himself that the external conditions are favorable, should not begin work unless he considers that he himself is in his normal condition. If he is not in that condition

he introduces an unknown disturbing element unnecessarily. He is also more liable to make mistakes in his readings and in his record. For the same reason he should not continue a series of observations too long at one time, as from fatigue the latter part of his work will not compare favorably with the first. In time-determinations, for instance, nothing is gained by observing from dark until daylight.

The observer is supposed to have no bias. A good observer, having taken all possible precautions with the adjustments of his instrument and knowing no reason for not doing good work, will feel a certain amount of indifference towards the results obtained. The man with a theory to substantiate is rarely a good observer, unless, indeed, he regards his theory as an enemy and not as a thing to be fondled and petted.

The greater an observer's experience the more do his habits of observation become fixed, and the more mechanical does he become in certain parts of his work. But his judgment may be constantly at fault. Thus with the astronomical transit he may estimate the time of a star crossing a wire in the focus of the telescope invariably too soon or invariably too late, according to the nature of his temperament. If he is doing comparison work involving micrometer bisections, he may consider the graduation mark sighted at to be exactly between the centre wires of the microscope when it is constantly on the same side of the centre. This fixed peculiarity, which none but experienced observers have, is known as their personal error.

In combining one observer's results with those of another observer we must either find by special experiment the difference of their personal errors and apply it as a correction to the final result, or else eliminate it by the method of observation. Thus in longitude work the present practice is to eliminate the effect of personal error from the final result by having the observers change places at the middle of the work.

It is always safer to eliminate the correction by the method of observing rather than by computing for it. For though it may happen that so long as instruments and conditions are the same the relative personal error of two observers may be constant, yet some apparently trifling change of conditions, such, for example, as illuminating the wires of the instrument differently, may cause it to be altogether changed in character.

On account of personal error, if for no other reason, it is evident that no number of sets of measures obtained in the same way by a single observer ought to be taken as furnishing a final determination of the value of a quantity. We must either vary the form of making the observations or else increase the number of observers, in the hope that personal error will eliminate itself in the final combination of the measures.

4. When all known corrections for instrument, for external conditions, and for peculiarities of the observer have been applied to a direct measure, have we obtained a correct value of the quantity measured? That we cannot say. If the observation is repeated a number of times with equal care different results will in general be obtained.

The reason why the different measures may be expected to disagree with one another has been indicated in the preceding pages. There may have been no change in the conditions of sufficient importance to have attracted the observer's attention when making the observations, but he may have handled his instrument differently, turned certain screws with a more or less delicate touch, and the external conditions may have been different. What the real disturbing causes were he has no means of knowing fully. If he had he could correct for them, and so bring the measures into accordance. Infinite knowledge alone could do this. With our limited powers we must expect a residuum of error in our best executed measures, and, instead of certainty in our results, look only for probability.

The discrepancies from the true value due to these un-

explained disturbing causes we call *errors*. These errors are *accidental*, being wholly beyond all our efforts to control. As soon as they are known to be *constant*, or we learn the law of their operation, they cease to be classed as errors.

A very troublesome source of discrepancies in measured values arises from *mistakes* made by the observer in reading his instrument or in recording his readings. Mistakes from imperfect hearing, from transposition of figures and from writing one figure when another is intended, from mistaking one figure on a graduated scale for another, as 7 for 9, 3 for 8, etc., are not uncommon. These also must be classed as accidental errors, theoretically at least.

Having, therefore, taken all possible precautions in making the observations and applied all known corrections to the observed values, the resulting values, which we shall in future refer to as the *observed values*, may be assumed to contain only accidental errors. We are, then, brought face to face with the question, How shall the value of the quantity sought be found from these different observed values?

Synopsis of Mathematical Principles Employed.

For convenience, and in order to avoid multiplicity of references, the leading principles of pure mathematics made use of in the further development of our subject are here placed together.

5. **Probability.**—(1) *The probability of the occurrence of an event is represented by the fraction whose denominator is the number of possible occurrences, all of which are supposed to be independent of one another and equally likely to happen, and whose numerator is the number of these occurrences favorable to the event in question.* Thus if an event may happen in a ways and fail in b ways, all equally likely to occur, the probability of its happening is $\frac{a}{a+b}$, and of its failing $\frac{b}{a+b}$, certainty being represented by unity.

(2) *If there are n events independent of each other, and the probability of the first happening is ψ_1, of the second ψ_2, and so on, the probability that all will happen is*

$$\psi_1 \cdot \psi \quad \ldots \quad \psi_n.$$

Thus if an urn contains two white and seven black balls the probability of drawing white at each of the first two trials, the ball not being replaced before the second trial, is

$$\frac{2}{9} \times \frac{1}{8} = \frac{1}{36}$$

6. **Definite Integrals.** — (a) To find the value of $\int_0^\infty e^{-t^2}dt$:

Let $t = xz$; then regarding x as a constant in integrating, we have

$$\int_0^\infty e^{-t^2}dt = \int_0^\infty e^{-x^2z^2}x\,dz$$

Multiply each member by $\int_0^\infty e^{-x^2}dx$, which is equal to $\int_0^\infty e^{-t^2}dt$, since the limits of integration are the same; then

$$\left\{\int_0^\infty e^{-t^2}dt\right\}^2 = \int_0^\infty dz \int_0^\infty e^{-x^2(1+z^2)}\,x\,dx$$

$$= \int_0^\infty \frac{dz}{2(1+z^2)}$$

$$= \frac{\pi}{4}$$

Hence $\int_0^\infty e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}$

(b) To find the value of $\int_{-\infty}^\infty at^2 e^{-a^2t^2}dt$:

Integrating by parts, we have

$$\int_{-\infty}^{\infty} at^2 e^{-a^2t^2} dt = \left(-\frac{te^{-a^2t^2}}{2a}\right)_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-a^2t^2}}{2a^2} d(at)$$

$$= \frac{\sqrt{\pi}}{2a^2}$$

Also, similarly,

$$\int_{-\infty}^{\infty} t^4 e^{-a^2t^2} dt = \frac{3\sqrt{\pi}}{4a^5}$$

(c) To find the value of $\int_0^a e^{-t^2} dt$:

The value of this integral cannot be expressed exactly in a finite form, but may be found approximately as follows:

Expanding e^{-t^2} in a series and integrating each term separately, we have

$$\int_0^a e^{-t^2} dt = \int_0^a \left(1 - \frac{t^2}{1} + \frac{t^4}{1.2} - \ldots\right) dt$$

$$= a - \frac{a^3}{3} + \frac{1}{1.2}\frac{a^5}{5} - \ldots$$

This series is convergent for all values of a; but the convergence is only rapid enough for small values of a.

For large values of a it is better to proceed as follows: Integrating by parts,

$$\int e^{-t^2} dt = \int -\frac{1}{2t} de^{-t^2}$$

$$= -\frac{1}{2t} e^{-t^2} - \frac{1}{2}\int \frac{e^{-t^2}}{t^2} dt$$

$$= -\frac{1}{2t} e^{-t^2} + \frac{1}{2^2t^3} e^{-t^2} + \frac{1.3}{2^2}\int \frac{e^{-t^2}}{t^4} dt$$

Hence

$$\int_a^{\infty} e^{-t^2} dt = \frac{e^{-a^2}}{2a}\left\{1 - \frac{1}{2a^2} + \frac{1.3}{(2a^2)^2} - \frac{1.3.5}{(2a^2)^3} + \ldots\right\}$$

But

$$\int_0^a e^{-t^2}dt = \int_0^\infty e^{-t^2}dt - \int_a^\infty e^{-t^2}dt$$

$$= \frac{\sqrt{\pi}}{2} - \int_a^\infty e^{-t^2}dt$$

$\therefore$ finally,

$$\int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a}\left\{1 - \frac{1}{2a^2} + \frac{1.3}{(2a^2)^2} - \frac{1.3.5}{(2a^2)^3} + \ldots\right\}$$

It is easily shown that, by stopping the summation at any term, the result will differ from the true value by less than the term stopped at.

Approximate values of the expression $\int_0^a e^{-t^2}dt$ may be computed from the above formulas for any numerical value of a.

7. **Taylor's Theorem.**—(a) If $f(x)$ is any function of x, and $f(x+h)$ is to be developed in ascending powers of h, then

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{1.2}f''(x) + \ldots \qquad (1).$$

A more rapid approximation is obtained by putting the development in the following form:

$$f\left(x+\frac{h}{2}+\frac{h}{2}\right) = f\left(x+\frac{h}{2}\right) + \frac{h}{2}f'\left(x+\frac{h}{2}\right) + \frac{1}{1.2}\left(\frac{h}{2}\right)^2 f''\left(x+\frac{h}{2}\right) + \ldots$$

$$f\left(x+\frac{h}{2}-\frac{h}{2}\right) = f\left(x+\frac{h}{2}\right) - \frac{h}{2}f'\left(x+\frac{h}{2}\right) + \frac{1}{1.2}\left(\frac{h}{2}\right)^2 f''\left(x+\frac{h}{2}\right) - \ldots$$

By subtraction and transposition,

$$f(x+h) = f(x) + h\,f'\left(x+\frac{h}{2}\right) + \frac{h^3}{24}f'''\left(x+\frac{h}{2}\right) + \ldots (2)$$

(b) Let F denote a function of a series of quantities X, Y, . . . expressed by the relation

$$F = f(X, Y, \ldots)$$

and let X', Y', . . . denote approximate values of X, Y, . . . and x, y, . . . the corrections to these approximate values, so that

$$\begin{aligned} X &= X' + x \\ Y &= Y' + y \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

then

$$\begin{aligned} F = f(X', Y', \ldots) &+ \frac{\partial F}{\partial X'} x + \frac{\partial F}{\partial Y'} y \\ &+ \frac{\partial^2 F}{\partial X'^2} x^2 + 2 \frac{\partial^2 F}{\partial X' \partial Y'} xy + \frac{\partial^2 F}{\partial Y'^2} y^2 \\ &+ \ldots \end{aligned}$$

where $\frac{\partial F}{\partial X'}$, $\frac{\partial F}{\partial Y'}$, . . . are the values found by differentiating $f(X, Y, \ldots)$, . . . with respect to X, Y, . . . and then substituting X', Y', . . . for X, Y, . . .

If the corrections x, y, . . . are so small that their squares and higher powers may be neglected and they are written dX', dY', . . . and $F - f(X', Y', \ldots)$ is written dF, then

$$dF = \frac{\partial F}{\partial X'} dX' + \frac{\partial F}{\partial Y'} dY' + \ldots$$

which is exactly the result found by differentiating F, which is a function of X, Y, . . . with respect to these quantities.

We shall use one form or the other as may be most convenient.

Ex. 1. If v is a very small correction to the number N, required to express $\log (N+v)$ in the linear form.

We have

$$\log (N+v) = \log N + \frac{d}{dN}(\log N)\ v + \ldots$$

$$= \log N + \frac{\text{mod.}}{N}\ v + \ldots$$

where mod. is the modulus of the common system of logarithms.

With a seven-place table we may use the formula

$$\log\ (N+v) = \log N + \delta_N\, v$$

where δ_N is the tabular difference corresponding to one unit for the number.

For small numbers, however, it is better to take δ_N from the table for the form

$$\tfrac{1}{2}\left\{\log (N+1) - \log (N-1)\right\}.$$

Thus from the table

$$\log\ (6543.2 + v) = 3.8157902 + 0.0000664\ v$$
$$\log\ (654.32 + v) = 2.8157902 + 0.0006637\ v$$
$$\log\ (65.432 + v) = 1.8157902 + 0.0066378\ v$$

Ex. 2. In a ten-place log. table where angles are given at regular intervals, required to find log sin $(A+a)$ when A is given in the table and a is a number of seconds less than the tabular interval.

We have

$$\log \sin (A+a) = \log \sin A + \frac{d}{dA}\left\{\log \sin \left(A+\frac{a}{2}\right)\right\} a$$

$$= \log \sin A + \text{mod. } \sin 1''\ a\ \cot \left(A + \frac{a}{2}\right)$$

Now,

$$\begin{aligned} \log \text{mod.} &= 9.6377843 - 10 \\ \log \sin 1'' &= 4.6855749 - 10 \\ \log 10^7 &= 7. \\ \hline & \quad 1.3233592 \end{aligned}$$

which expresses log (mod. sin $1''$) in terms of the seventh place of decimals as the unit. Hence

$$\log \sin (A+a)$$
$$= \log \sin A + \log^{-1}\left\{1.3233592 + \log a + \left(\log \cot A + \frac{a}{2} \times \text{diff. for } 1''\right)\right\}$$

With Vega's Thesaurus,* which gives the log. functions to single seconds to the end of the first degree and afterwards for every $10''$, log sin $(A+a)$ can be found from the above expression for values of $A > 3°$, and also when A lies between $20'$ and $2°$ to within less than unity in the tenth decimal place. Between $2°$ and $3°$ the difference may be as large as 3 units in the tenth place from the value found by carrying out the formula more exactly. But in the

* *Thesaurus Logarithmorum Completus.* Lipsiæ, 1794.

Thesaurus, in the trigonometrical part, "the uncertainty of the last figure amounts to 4 units."* Hence the above process is in general sufficient when this table is used.

With a seven-place table of log. sines it would be, in general, sufficient to take the tabular difference δ_A for 1″ for the angle A as the value of

$$\frac{d}{dA}\left\{\log\sin\left(A+\frac{a}{2}\right)\right\}$$

so that

$$\log\sin(A+a)=\log\sin A+\delta_A\, a.$$

Ex. 3. If A is the approximate value of an angle, and v a correction to it so small that its square and higher powers may be neglected, required to express log sin $(A+v)$ in the linear form, using a ten-place table.

Let A_1 be the angle nearest to A in the table, and set

$$A=A_1+a$$

then

$$\begin{aligned}
&\log\sin(A+v)\\
&=\log\sin(A_1+a+v)\\
&=\log\sin A_1+\text{mod.}\sin 1''\ a\cot\left(A_1+\frac{a}{2}\right)+\text{mod.}\sin 1''\cot(A_1+a)\,v\\
&=\log\sin A_1+\log^{-1}\left\{1.3233592+\log a+\left(\log\cot A_1+\frac{a}{2}\times\text{diff. for }1''\right)\right\}\\
&\qquad+\log^{-1}\left\{1.3233592+(\log\cot A_1+a\times\text{diff. for }1'')\right\}v
\end{aligned}$$

Expand log sin $(68°\ 16'\ 32''.076+v)$

$$A_1=68°\ 16'\ 30''$$
$$a=\quad 2''.076$$

	1.3233592	
log cot A_1	9.6003780	
	0.9237372	
$a\times$diff. 1″	—127	
	0.9237245	8,3893
$-\frac{a}{2}\times$diff. 1″	+64	
	0.9237309	
log a	0.3172273	
	1.2409582	17,416
log sin 68° 16′ 30″		9.9680022,271

Hence log sin $(68°\ 16'\ 32''.076+v)=$ 9.9680039,687 + 8,3893 v

when the difference is expressed in terms of the seventh decimal place as the unit.

* Bremiker's edition of Vega, translated by Fischer. Preface, p. 10.

With a seven-place table, except for small angles or angles near 180° it will be sufficient to take

$$\log \sin (A_1+a+v)=\log \sin A_1+\delta_A\, a+\delta_A\, v$$

when δ_A is the tabular difference corresponding to 1″ for the angle A_1. It can be taken by inspection from the table. Thus,

$$\log \sin (68°\ 16'\ 32''+v)=9.9680039+8{,}4\ v$$

8. **Interpolation.**—So far as interpolation is concerned, we have mainly to deal with the logarithms of trigonometric functions. The differences between the successive values given in a table are *first differences*, and the differences between the successive first differences are *second differences*. Beyond second differences we do not need to go.

This may be expressed in tabular form :

Function.	First diff.	Second diff.
$f(A)$		
	d_1	
$f(A+a)$		d_2
	d_1'	
$f(A+2a)$		

Hence

$$\begin{aligned} f(A+a) &= f(A)+d_1 \\ f(A+2a) &= f(A+a)+d_1' \\ &= f(A)+2d_1+d_2 \end{aligned}$$

Generally

$$f(A+na)=f(A)+nd_1+\frac{n(n-1)}{2}\,d_2+\ldots$$

which is Newton's formula.

Ex. In a ten-place table of log. sines in which values are

given to every 10 seconds, required log sin A when A is any angle.

Let A_1 = the part of the given angle A to the nearest 10 seconds that occurs in the table.

a = the units of seconds and parts of seconds in the given angle.

d_1, d_2 = the first and second tabular differences.

then

$$\log \sin A = \log \sin \left(A_1 + \frac{a}{10} \cdot 10\right)$$

$$= \log \sin A_1 + \frac{a}{10} d_1 + \tfrac{1}{2} \frac{a}{10} \left(\frac{a}{10} - 1\right)(-d_2)$$

$$= \log \sin A_1 + a \frac{d_1}{10} + \tfrac{1}{2} a (10 - a) \frac{d_2}{100}. \qquad (1)$$

Writing this in the form,

$$\log \sin A = \log \sin A_1 + a \left\{ \frac{d_1}{10} + \left(5 - \frac{a}{2}\right) \frac{d_2}{100} \right\} \quad (2)$$

we have the convenient rule: Assume the second difference constant throughout the interval a. Then from the first and second differences find by simple interpolation the value of the difference at the middle of the interval. This difference multiplied by the interval gives the correction to the tabular log. sine.

To find log sin 68° 16′ 32″.076:

From the table,

log sin 68° 16′ 30″ = 9.9680022,271

d_1 = 83,889 for 10″ in units of the seventh decimal place

d_2 = 0,012

Hence, from equation (1),

$$\text{Corr. to tab. value} = 2.076 \times 8,3889 + \frac{2.076}{2}(10 - 2.076)\frac{12}{100}$$

$$= \quad 17,416$$

∴ log sin 68° 16′ 32″.076 = 9.9680039,687

The difference at $35'' = 8{,}3889$.
We want it at $31''.038$, the middle of the interval.
Now, change of first difference $= 0{,}00012$ for $1''$.

$$\begin{aligned}
\text{Hence corr. to first difference} &= (35 - 31.038) \times 0{,}00012 \\
&= 0{,}0005 \\
\text{First diff.} &= 8{,}3889 \\
\hline
\text{Diff. required} &= 8{,}3894
\end{aligned}$$

And $2.076 \times 8{,}3894 = 17{,}416$ as before.

9. **Periodic Series.**—To sum the series

$$\text{Cos } o + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta$$
$$\text{Sin } o + \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta$$

where $\theta = \frac{360}{n}$, n being an integer, we may proceed as follows:

If θ is the angle which a line $B'OB$ makes with OA then the projection of OB on OA is $OB \cos\theta$, and the projection of OB' is $-OB\cos\theta$ if $OB' = OB$. The projections on a line at right angles to OA are $OB \sin\theta$ and $-OB\sin\theta$ respectively.

If we divide the circumference of a circle into n equal parts at the points $A, B, \ldots$ then each angle at the centre O is $\frac{360°}{n}$, or θ, and by projecting the lines $OA, OB, \ldots$ on the diameter through A we find the sum of the first series to be zero, and by projecting the same lines on the diameter perpendicular to OA we find the sum of the second series to be also zero.

These results may be written:

$$\Sigma \cos m\theta = o \qquad\qquad \Sigma \sin m\theta = o$$

where m assumes all values from o to $n - 1$.

Hence it follows that

$$\Sigma \sin m\theta \cos m\theta = \tfrac{1}{2}\,\Sigma \sin m\,2\theta = o$$

$$\Sigma \cos^2 m\theta \qquad = \frac{n}{2} + \tfrac{1}{2}\Sigma \cos m\,2\theta = \frac{n}{2}$$

$$\Sigma \sin^2 m\theta \qquad = \frac{n}{2} - \tfrac{1}{2}\Sigma \cos m\,2\theta = \frac{n}{2}$$

10. **Notation.**—The following convenient notation, introduced by Gauss, is now very generally used in the method of least squares.

If $a_1, a_2, \ldots$ are quantities of the same kind, their algebraic sum is denoted by $[a]$, and the sum of their squares by $[aa]$ or $[a^2]$, so that

$$[a] = a_1 + a_2 + \ldots + a_n$$
$$[aa] \text{ or } [a^2] = a_1^2 + a_2^2 + \ldots + a_n^2$$

Also,

$$[ab] = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$
$$\left[\frac{ab}{c}\right] = \frac{a_1 b_1}{c_1} + \frac{a_2 b_2}{c_2} + \ldots + \frac{a_n b_n}{c_n}$$

We shall use the symbol $[a$ to denote the sum of a series of quantities a all taken with the same sign.

CHAPTER II.

THE LAW OF ERROR.

The Arithmetic Mean.

11. (*a*) **When the quantity measured is the quantity to be found.**—If $M_1, M_2 \ldots M_n$ are n direct and independent measures of a quantity, T, we may write

$$\left.\begin{array}{l} T-M_1=\Delta_1 \\ T-M_2=\Delta_2 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ T-M_n=\Delta_n \end{array}\right\} \qquad (1)$$

where $\Delta_1, \Delta_2, \ldots \Delta_n$ indicate the differences between T and the observed values, and are therefore the errors of observation.

We have here n equations and $n+1$ unknowns. What principle shall we call to our aid to solve these equations and so find $T, \Delta_1, \Delta_2, \ldots \Delta_n$? In answering this question I shall follow the order of natural development of the subject, which, in the main, is also the order of its historical development.

The value sought must be some function of the observed values and fall between the largest and smallest of them. If the observed values are arranged according to their magnitudes they will be found to cluster around a central value. On first thoughts the value that would be chosen as the value of T would be the central value in this arrangement if the number of observations were odd, and either of the two central values if the number were even. In other words, a plausible value of the unknown would be that observed value which had as many observed values greater than it as it had less than it. Now, since a small change in any of the observed values, other than the central

value, would in general produce no change in the result, the number of observations remaining the same, this method of proceeding might be regarded as giving a plausible result, more especially if the observed values were widely discrepant.

On the other hand, the taking of the central value is objectionable, because it gives the preference to a single one of the observed values, while if these values are supposed to be equally worthy of confidence, as it is reasonable to take them in the absence of all knowledge to the contrary, each ought to exert an equal influence on the result. We may, therefore, with more reason assume the value of T to be a *symmetrical function* of the observed values.* Also, since a change in the number of observations should produce no change in the form of this function, it follows that the function must be of such a form as to satisfy the condition that when the observed values are all equal to one another it will reduce to this common value; that is, if $T = f(M_1\, M_2 \; . \; . \; . \; M_n)$, and $M_1 = M_2 = \; . \; . \; . \; M_n = M$, then $f(M, M, \; . \; . \; . \; M) = M$. For if we had a single observation, then necessarily $f(M) = M$.

Let, then, V be a symmetrical function of $M_1\, M_2 \; . \; . \; . \; M_n$, and put

$$
\begin{aligned}
V - M_1 &= v_1 \\
V - M_2 &= v_2 \qquad\qquad (2) \\
. \quad . \quad &. \quad . \quad . \\
V - M_n &= v_n
\end{aligned}
$$

Expanding by Taylor's theorem:

$$
\begin{aligned}
V &= f(M_1\, M_2 \; . \; . \; . \; M_n) \\
&= f(V - v_1,\, V - v_2, \; . \; . \; . \;) \\
&= f(V, V, \; . \; . \; .) - [v]\frac{df}{dv} + H + K + \; . \; . \; . \qquad (3)
\end{aligned}
$$

where $H, K, \; . \; . \; . \; .$ are terms involving the second, third, . . . powers of the small quantities, $v_1, v_2, \; . \; . \; . \; v_n$

* See Reuschle, *Crelle Jour. Math.*, vol. xxvi.; Schiaparelli, *Rendiconti del R. Instituto Lombardo*, 1868; *Astron. Nachr.*, 2068, 2097; Stone, *Month. Not. Roy. Astron. Soc.*, vol. xxxiii.; *Astron. Nachr.*, 2092; Ferrero, *Expos. del Met. dei Min. Quadr.*, Florence, 1876. Also Fechner, *Ueber den Ausgangswerth der kl. Abweichungssumme*, Leipzig, 1874.

The simplest symmetrical function of the observed values that can be chosen as the form for V is their arithmetic mean—that is, $\frac{[M]}{n}$. If we take V equal to this value, then from equation (1) by addition $[v]=0$, and, neglecting powers of v higher than the first, equation (3) is satisfied identically. Thus far, therefore, it would appear that the arithmetic mean may be taken as one solution. It may happen that the values $M_1, M_2, \ldots M_n$ are of such a nature that some other symmetrical function than the arithmetic mean will satisfy (2) better than will the arithmetic mean. That the arithmetic mean is *on the whole* the best form for the function $f(M_1, M_2, \ldots M_n)$, when $M_1, M_2, \ldots M_n$ are direct measures of some phenomenon in the sciences of observation, which sciences only we intend to consider, may be confirmed by a comparison of results flowing from this hypothesis with the records of experience. This we shall do later (see Art. 24, 37, 51).

The older mathematicians, as Cotes and Simpson, laid the foundations of our subject by announcing the principle of the arithmetic mean. Gauss, to whom we owe the first complete exposition, assumed the arithmetic mean as a plausible hypothesis;* and Hansen, who made the next great advances, started from it as an axiom. The principle itself may be stated as follows: If we have n observed values of an unknown, all equally good so far as we know, the most plausible value of the unknown (best value on the whole) is the arithmetic mean of the observed values.

12. By adding equations (1), Art. 11, and taking the mean, we have

$$T=\frac{[M]}{n}+\frac{[\Delta]}{n}$$
$$=V+\frac{[\Delta]}{n}$$

The last term of this equation will become very small if,

* Gauss' words are, "Axiomatis loco haberi solet hypothesis." (*Theoria Motus*, lib. 2, sec. 3.)

n being very large, the sum $[\Delta]$ of the errors remains small. Now, if, after making one observation and before making another, we readjust our instrument, determine anew its corrections, choose the most favorable conditions for observing, and vary the form of procedure as much as possible, it is reasonable to suppose that the disturbing influences will balance one another in the result following from the proper combination of the observed values. It may take an infinite number of trials to bring this about. In the absence of all knowledge we cannot say that it will take less. And, reckoning mistakes in reading the instrument or in recording the readings as accidental errors, an infinity of a higher order than the first may be required to eliminate them.

In other words, there being no reason to suppose that an error in excess (or positive error) is more likely to occur on the whole than an error in defect (or negative error), we may, when n is a very large number, consider $\frac{[\Delta]}{n}$ to be an infinitesimal with respect to T. We may, therefore, in this case put

$$V = T$$

—that is, *when the number of observed values is very great the arithmetic mean is the true value.*

13. From the principle of the arithmetic mean two important inferences may be derived. For, taking the arithmetic mean, V, of n observed values of an unknown as the most plausible value of that unknown, we may write our *observation-equations* in the form,

$$\left.\begin{array}{l} V - M_1 = v_1 \\ V - M_2 = v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ V - M_n = v_n \end{array}\right\} \qquad (1)$$

where $v_1, v_2, \ldots v_n$ are called the *residual errors* of observation, or simply the *residuals*.

(a) By addition

$$nV-[M]=[v]$$

and $\therefore$

$$[v]=0. \tag{2}$$

—that is, *the sum of the residuals is zero;* in other words, *the sum of the positive residuals is equal to the sum of the negative residuals.*

There is a very marked correspondence between the series in which n is infinitely great and T is the *true* value, and a series in which n is finite and the arithmetic mean V is taken as the best value attainable. For in the first case the sum of the *errors*, Δ, divided by n, is zero, and in the second the sum of the *residuals*, v, is zero.

(b) Let V' be any assumed value of the unknown other than the arithmetic mean, and put

$$\left.\begin{array}{l} V'-M_1=v'_1 \\ V'-M_2=v'_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ V'-M_n=v'_n \end{array}\right\} \tag{3}$$

From equations (1) and (3), by squaring and adding,

$$[vv]=nVV-2V[M]+[MM]$$
$$[v'v']=nV'V'-2V'[M]+[MM]$$

Hence by a simple reduction,

$$[v'v']=[vv]+n\left(V'-\frac{[M]}{n}\right)^2$$

Now, $\left(V'-\frac{[M]}{n}\right)^2$, being a complete square, is always positive.

$$\therefore [v'v'] > [vv]$$

—that is, *the sum of the squares of the residuals v, found by taking the arithmetic mean, is a minimum.*

Hence the name *Method of Least Squares*, which was first given by Legendre.

Let us recapitulate the three forms of solution proposed

for finding the most plausible value V of the unknown from the n equations:

$$V - M_1 = v_1$$
$$V - M_2 = v_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V - M_n = v_n$$

(1) Find all possible values of V and take the mean. The values of V are M_1, M_2, . . . M_n, since for each observation considered singly the best value must be the directly observed one, and the mean of these values is $\frac{[M]}{n}$.

(2) Solve simultaneously, making $[v] = 0$. This gives

$$nV - [M] = [v] = 0$$

$$\text{or} \qquad V = \frac{[M]}{n}$$

the arithmetic mean.

(3) Solve simultaneously, making $[vv] =$ a minimum. This gives

$$(V - M_1)^2 + (V - M_2)^2 + \ . \ . \ . \ (V - M_n)^2 = \text{a min.}$$

Differentiating with respect to V,

$$V - M_1 + V - M_2 + \ . \ . \ . \ V - M_n = 0$$

$$\text{or} \qquad V = \frac{[M]}{n}$$

the arithmetic mean.

14. **When the quantity measured is a function of the quantities to be found.**—We pass now to the more general but equally common case in which the observations, instead of being made directly on the quantity to be determined, are made indirectly—that is, made on a quantity which is a function of the quantities whose values are to be found.

Thus, let the function connecting the observed quantity T and the unknowns $X, Y, \ldots$ be

$$T = f(X, Y, \ldots) \qquad (1)$$

in which the constants involved are given by theory for each observation.

If $M_1, M_2, \ldots$ are the observed values of T, the equations for finding the unknowns, when reduced to the linear form, may be written

$$\left.\begin{array}{l} v_1 + M_1 = a_1 X + b_1 Y + \ldots - L_1 \\ v_2 + M_2 = a_2 X + b_2 Y + \ldots - L_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \qquad (2)$$

in which $a_1, b_1, \ldots L_1, \ldots$ are known constants, and $v_1, v_2, \ldots$ are the residual errors of observation.

Now, if the number of equations, n, is equal to the number of unknowns, n_i, the values of $X, Y, \ldots$ may be found by the ordinary algebraic methods, and if substituted in the equations will satisfy them exactly. But if the number of equations exceeds the number of unknowns, the values found from a sufficient number of the equations will not in general satisfy the remaining equations exactly. Many such sets of values may be found, which are therefore all *possible* solutions. But of all these possible sets some one will satisfy the equations better than any of the others. We have so far no means of knowing when we have found this *most plausible* (best on the whole) set of values. With a single unknown the arithmetic mean gives the most plausible result. Let us see if a method of finding means corresponding to those of Art. 13 will apply to these equations.

For simplicity in writing take the three equations:

$$\begin{array}{l} a_1 X + b_1 Y = M_1 + v_1 \\ a_2 X + b_2 Y = M_2 + v_2 \\ a_3 X + b_3 Y = M_3 + v_3 \end{array}$$

where $a_1, b_1, \ldots$ are known, and X, Y are to be found.

(1) Find all possible values of X and Y, and combine them.

To do this we form all possible sets of two equations and solve each set. Thus,

$$\begin{array}{lll} a_1X+b_1Y=M_1 & a_1X+b_1Y=M_1 & a_2X+b_2Y=M_2 \\ a_2X+b_2Y=M_2 & a_3X+b_3Y=M_3 & a_3X+b_3Y=M_3 \end{array}$$

whence at once

$$(a_1b_2-a_2b_1)X=b_2M_1-b_1M_2$$
$$(a_1b_2-a_2b_1)Y=a_1M_2-a_2M_1$$

$$(a_1b_3-a_3b_1)X=b_3M_1-b_1M_3$$
$$(a_1b_3-a_3b_1)Y=a_1M_3-a_3M_1$$

$$(a_2b_3-a_3b_2)X=b_3M_2-b_2M_3$$
$$(a_2b_3-a_3b_2)Y=a_2M_3-a_3M_2.$$

In combining these values of X and of Y we are met by a difficulty. It would not do to take the arithmetic means as the most plausible values, for X and Y may be better determined from one set of equations than from another, and the arithmetic mean gives the most plausible value only on the assumption that all of the values combined in it are of equal quality. It is necessary, therefore, to have a method of combining observations of different quality before we can find X and Y in this way.

(2) The simultaneous solution of the equations by making the sum of the residuals equal to zero.

Hence X, Y should be found from

$$a_1X+b_1Y-M_1+a_2X+b_2Y-M_2+a_3X+b_3Y-M_3=0$$

which is impossible, as this equation may be satisfied by an infinite number of values of X and Y. The principle is, therefore, insufficient.

(3) The simultaneous solution by making the sum of the squares of the residuals a minimum. We have

$$(a_1X+b_1Y-M_1)^2+\ldots+\ldots=\text{a min.}$$

Differentiating with respect to X, Y as independent variables, we find

$$[aa]\,X+[ab]\,Y=[aM]$$
$$[ab]\,X+[bb]\,Y=[bM]$$

where

$$[aa]=a_1a_1+a_2a_2+a_3a_3$$
$$[ab]=a_1b_1+a_2b_2+a_3b_3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

This method gives as many equations as unknowns, and so but one set of values of X and Y can be found.

We cannot, however, say that we have found the most plausible values of X and Y. All we can say is that the last method employed reduces the number of equations to the number of unknowns and gives us one set of values of X and Y, and that the same principle applied to the special case of one unknown gives the most plausible value of that unknown, in that it gives the arithmetic mean. Analogy, however, would lead us to suspect that we have found the most plausible values of X and Y. With one unknown, if the separate observed values, represented by lines of the proper length, are plotted along a straight line from a certain assumed origin, and equal weights are placed at the end points, the position of the centre of gravity of the weights will coincide with the end of the line representing the arithmetic mean of the distances, and the sum of the squares of the distances of the weights from the centre of gravity is a minimum.

Again, the centre of gravity of n equally well-observed positions of a point in space would be the most plausible mean position to take for the point. But it is a well-known principle that, if equal weights be placed at n points in space, the centre of gravity of these weights is at a point such that the sum of the squares of its distances from the weights is a minimum.* On this principle Legendre founded the rule of minimum squares, and he employed the rule

* See, for example, Todhunter's *Statics*, Art. 138.

as giving a *convenient* method of solution in the class of problems under consideration.

The Law of Error of a Single Observed Quantity.

15. With a single unknown we have seen that the most plausible value is the arithmetic mean of the independently observed values, and that it can be found by making the sum of the squares of the residuals a minimum. The methods are equivalent.

With more than one unknown we have failed to find this correspondence of methods. The reasoning from analogy in the preceding article is well enough as far as it goes, but it is not conclusive. The difficulty lies in combining values not equally good. We must, therefore, devise some method of combining such values before a rule for finding means can be applied to several unknowns.

Now, when several independent measures of the same quantity, all equally good, have been made, it must be granted that errors in excess and errors in defect are equally likely to occur to the same amount—that is, are equally probable. Experience shows that in any well-made series of observations small errors are likely to occur more frequently than large ones, and that there is a limit to the magnitude of the error to be expected. If, therefore, a denotes this limit or maximum error, we must consider all the errors of the series to be ranged between $+a$ and $-a$, but to be most numerous in the neighborhood of zero. Hence the probability of the occurrence of an error may be assumed to be a certain function of the error.

If, then, the probability that an error does not exceed Δ be denoted by $\xi(\Delta)$, the probability of an error between Δ and $\Delta + d\Delta$ is

$$\begin{aligned} \xi(\Delta + d\Delta) - \xi(\Delta) &= \xi'(\Delta)\, d\Delta \\ &= \varphi(\Delta)\, d\Delta \quad \text{suppose.} \qquad (1) \end{aligned}$$

The function $\varphi(\Delta)$ is called the law of distribution of error, or simply the *law of error*.

The probability that an error falls between any assigned limits b and a is the sum of the probabilities $\varphi(\Delta)\,d\Delta$ extending from b to a, and is expressed in the ordinary notation of the integral calculus by

$$\int_a^b \varphi(\Delta)\,d\Delta \qquad (2)$$

Hence it follows that the probability that an error does not exceed the value a is

$$\int_{-a}^{+a} \varphi(\Delta)\,d\Delta \qquad (3)$$

The properties of errors stated above are not sufficient to determine the form of the function $\varphi(\Delta)$ definitely. Among other forms that might be chosen to satisfy them the simplest would be

$$\varphi(\Delta) = e^{-h\Delta} \qquad (4)$$

where h is a constant.

This was, indeed, that selected by Laplace in his investigation *Determiner le milieu que l'on doit prendre entre trois observations données d'un même phénomène* (Mém. Acad. Paris, 1774). From this form may be readily derived the results stated in Art. 11.

The form of $\varphi(\Delta)$ may, however, be more satisfactorily determined by calling in the aid of the calculus of probabilities. For if $\varphi(\Delta_1)\,d\Delta_1$, $\varphi(\Delta_2)\,d\Delta_2$, . . . denote the probabilities of the occurrence of errors in n observations between Δ_1 and $\Delta_1 + d\Delta_1$, Δ_2 and $\Delta_2 + d\Delta_2$, . . . respectively, the probability of the simultaneous occurrence of this system of errors is proportional to the product (see Art. 5).

$$\varphi(\Delta_1)\,\varphi(\Delta_2)\,\ldots\,\varphi(\Delta_n)$$

Denote this expression by ψ, so that

$$\psi = \varphi(\Delta_1)\,\varphi(\Delta_2)\,\ldots\,\varphi(\Delta_n) \qquad (5)$$

Now, the true value T, and therefore the values of Δ_1, Δ_2, . . . Δ_n, are unknown. If we make the expression for ψ a

maximum, we should find the most probable value of the unknown. But we have seen that the most plausible value of the unknown is the arithmetic mean of the observed values, and that when the number of observations is very large the arithmetic mean is the true value T. Calling, then, the most plausible value the most probable value, we have, when n is large, the true value by making ψ a maximum. The form of the function φ will, therefore, result from this hypothesis.

Now, since $\log \psi$ varies as ψ, we must have $\log \psi$ a maximum, and therefore by differentiation

$$0=\frac{d \log \psi}{dT}=\frac{\partial \log \varphi(\Delta_1)}{\partial \Delta_1}\frac{d\Delta_1}{dT}+\frac{\partial \log \varphi(\Delta_2)}{\partial \Delta_2}\frac{d\Delta_2}{dT}+\dots$$

or

$$0=\frac{1}{\psi}\frac{d\psi}{dT}=\Delta_1\frac{\partial \log \varphi(\Delta_1)}{\Delta_1\,\partial \Delta_1}+\Delta_2\frac{\partial \log \varphi(\Delta_2)}{\Delta_2\,\partial \Delta_2}+\dots \qquad (6)$$

since from equation (1), Art. 11,

$$\frac{d\Delta_1}{dT}=\frac{d\Delta_2}{dT}=\dots=\frac{d\Delta_n}{dT}=1$$

But, from the principle of the arithmetic mean, when the number of observations is very great,

$$\Delta_1+\Delta_2+\dots+\Delta_n=0. \qquad (7)$$

Also, since equations (6) and (7) must be simultaneously satisfied by the same value of the unknown, being the most probable value in either case, and since the errors $\Delta_1, \Delta_2, \dots \Delta_n$ are connected only by the relation $[\Delta]=0$, we necessarily have, when $n>2$,*

$$\frac{\partial \log \varphi(\Delta_1)}{\Delta_1\,\partial \Delta_1}=\frac{\partial \log \varphi(\Delta_2)}{\Delta_2\,\partial \Delta_2}=\dots=k \text{ suppose.}$$

Clearing of fractions and integrating,

$$\varphi(\Delta)=ce^{\frac{1}{2}k\Delta^2}.$$

* When $n=2$ it reduces to an identity.

where e is the base of the Napierian system of logarithms and c is a constant.

Now, since ψ is to be a maximum, $\frac{d^2\psi}{dT^2}$ must be negative. But when ψ is a maximum subject to the condition $[\Delta]=0$, then $\frac{d^2\psi}{dT^2}=nk\psi$. Hence, since ψ is positive, k must be negative, and, putting it equal to $-\frac{1}{\mu^2}$, we have

$$\varphi(\Delta)=ce^{-\frac{\Delta^2}{2\mu^2}}$$

the law of error sought.

16. In this expression there are two symbols undetermined, c and μ. To find c. Since it is certain that all of the errors lie between the maximum errors $+a$ and $-a$, we have

$$c\int_{-a}^{+a}e^{-\frac{\Delta^2}{2\mu^2}}d\Delta=1$$

But as the values of a are different for different kinds of observations, and as we cannot in general assign these values definitely, we must take $+\infty$ and $-\infty$ as the extreme limits of error, so that c is found from

$$c\int_{-\infty}^{+\infty}e^{-\frac{\Delta^2}{2\mu^2}}d\Delta=1$$

and hence (see Art. 6)

$$c=\frac{1}{\mu\sqrt{2\pi}}$$

and the law of error may be written

$$\varphi(\Delta)=\frac{1}{\mu\sqrt{2\pi}}\,e^{-\frac{\Delta^2}{2\mu^2}}$$

or by putting $h^2=\frac{1}{2\mu^2}$

$$\varphi(\Delta)=\frac{h}{\sqrt{\pi}}\,e^{-h^2\Delta^2}$$

When this latter form is used it is only for greater convenience in writing, and h is to be looked on as a mere symbol standing for $\frac{1}{2\mu^2}$.

As regards μ^2, it is evident that for $e^{-\frac{\Delta^2}{2\mu^2}}$ to be a possible quantity $\frac{\Delta^2}{\mu^2}$ must be an abstract number. Hence μ is a quantity expressed in the same unit of measure as Δ.

Also, from the form of the function $\varphi(\Delta)$, it is evident that the probability of an error Δ will be the larger the larger μ is, and *vice versa*. Hence μ is a test of the quality of observations of different series, the unit being the same.

Again, the total number of errors in a series being n, the number between Δ and $\Delta + d\Delta$ will, from the definition of probability, be $n\,\varphi(\Delta)\,d\Delta$. Hence the sum of the squares of the errors Δ in the same interval will be equal to $n\Delta^2\,\varphi(\Delta)\,d\Delta$, and the sum of the squares of the errors between the limits of error $+a$ and $-a$ will be

$$n\int_{-a}^{+a}\Delta^2\,\varphi(\Delta)\,d\Delta$$

Extending the limits of error $\pm a$ to $\pm\infty$, this expression becomes, after substituting for $\varphi(\Delta)$ its value,

$$\frac{n}{\mu\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Delta^2\,e^{-\frac{\Delta^2}{2\mu^2}}\,d\Delta$$

which (see Art. 6) reduces to $n\mu^2$.

Hence μ^2 is the mean of the sum of the squares of the errors Δ that occur in the series. It is called the *mean-square error*, and will be referred to as the m. s. e.

17. **The Principle of Least-Squares.**—Having defined the symbols c and μ in the expression for $\varphi(\Delta)$, let us return to Art. 15, Eq. 5.

If the observed values are of the same quality throughout, μ is constant and the product ψ becomes $c^n e^{-\frac{[\Delta^2]}{2\mu^2}}$. This

product is evidently a maximum when $[\Delta^2]$ is a minimum; that is, *if we assume that each of a very large number of observed values of a quantity is of the same quality, the most probable value of the quantity is found by making the sum of the squares of the errors a minimum.*

If the observed values are not of the same quality, μ is different for the different observations, and the most probable value of the unknown would be found from the maximum value of $e^{-\left[\frac{\Delta^2}{2\mu^2}\right]}$; that is, from the minimum value of $\left[\frac{\Delta^2}{\mu^2}\right]$. Thus *if each of a large number of observed values of a quantity is of different quality, the most probable value of the quantity is found by dividing each error of observation by its m. s. e. and making the sum of the squares of the quotients a minimum.*

This latter includes the case in which the observed quantity is a function of several independent unknowns whose values are to be found. For if

$$M_1 = f_1(x, y, \ldots)$$
$$M_2 = f_2(x, y, \ldots)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

where the functions $f_1, f_2, \ldots$ are of the same form and differ only in the constants that enter; then if $\Delta_1, \Delta_2, \ldots$ denote the errors of observation, we have

$$\Delta_1 = f^1(x, y, \ldots) - M_1$$
$$\Delta_2 = f^2(x, y, \ldots) - M_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence since $\Delta_1, \Delta_2, \ldots$ are functions of the independent variables $x, y, \ldots$ we must have

$$\left[\frac{1}{\mu^2}\{f(x, y, \ldots) - M\}^2\right] = \text{a minimum}$$

with respect to the variables $x, y, \ldots$ But the differentiation of this equation with respect to $x, y, \ldots$ and the equating of the differential coefficients to zero, gives as

many equations as unknowns, from which equations the most probable values of x, y, . . . may be found.

18. Two other inferences from the preceding general principles are important:

(a) Since in a series of observed values of different quality the sum of the squares of the errors, divided by their respective m. s. e. made a minimum, leads to the most probable values, it follows that observed values of different quality are put on a common basis for combination by dividing by their respective m. s. e. This conclusion will be found developed in Chapter III.

(b) Since $\frac{\Delta^2}{\mu^2}$ is an abstract number, no matter what the unit of measure in which the observed values are expressed, it follows that *heterogeneous measures* may be combined in the same minimum equation. For an example in which this is fully brought out see Art. 169.

The Law of Error of a Linear Function of Independently Observed Quantities.

19. We have found the law of error in the case of a quantity directly observed, and which may be a function of one or more unknowns. There remains the question as to the form the law of error assumes in the case of a quantity, F, which is a linear function of several independently observed quantities, $M_1, M_2, \ldots M_n$; that is, when

$$F = a_1M_1 + a_2M_2 + \ldots + a_nM_n$$

where $a_1, a_2, \ldots$ are constants.

For simplicity in writing consider two observed quantities, M_1, M_2, only, and let μ_1, μ_2 be their m. s. e. The probability of the simultaneous occurrence of the errors Δ_1 in M_1 and Δ_2 in M_2 is

$$\frac{h_1 h_2}{\pi} e^{-h_1^2\Delta_1^2 - h_2^2\Delta_2^2} d\Delta_1\, d\Delta_2 \qquad (1)$$

Now, an error Δ_1 in M_1 and an error Δ_2 in M_2 produce an error Δ in F, according to the relation

$$\Delta = a_1\Delta_1 + a_2\Delta_2 \qquad (2)$$

and this relation can always be satisfied by combining any value of Δ_2 with all values of Δ_1 ranging from $-\infty$ to $+\infty$. The probability, therefore, of an error Δ in F may be written

$$\varphi(\Delta)\, d\Delta = \frac{h_1 h_2}{\pi}\, d\Delta_2 \int_{-\infty}^{+\infty} e^{-h_1^2\Delta_1^2 - h_2^2\Delta_2^2} d\Delta_1$$

But from (2), and since Δ_2 is independent of Δ_1,

$$d\Delta = a_2\, d\Delta_2$$

Hence

$$\varphi(\Delta)\, d\Delta$$

$$= \frac{h_1 h_2}{\pi}\frac{d\Delta}{a_2}\int_{-\infty}^{+\infty} e^{-h_1^2\Delta_1^2 - h_2^2\left(\frac{\Delta - a_1\Delta_1}{a_2}\right)^2} d\Delta_1$$

$$= \frac{h_1 h_2}{\pi}\frac{d\Delta}{a_2} e^{-\frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2^2 a_1^2}\Delta^2} \int_{-\infty}^{+\infty} e^{-\frac{h_1^2 a_2^2 + h_2^2 a_1^2}{a_2^2}\left(\Delta_1 - \frac{h_2^2 a_1}{h_1^2 a_2^2 + h_2^2 a_1^2}\Delta\right)^2} d\Delta_1$$

$$= \frac{h_1 h_2}{\sqrt{(h_1^2 a_2^2 + h_2^2 a_1^2)\pi}} e^{-\frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2^2 a_1^2}\Delta^2} d\Delta$$

which is of the form

$$\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta$$

That is, *the law of error of the function F is the same as that of the independently measured quantities M_1, M_2.*

The m. s. e. of the function F is found from

$$h^2 = \frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2^2 a_1^2}$$

that is, from

$$\mu^2 = a_1^2\mu_1^2 + a_2^2\mu_2^2$$
$$= [a^2\mu^2]$$

This theorem is one of the most important in the method of least squares, and will be often referred to.

Ex. To find the m. s. e. of the arithmetic mean of n equally well observed values of a quantity:

We have

$$F = \frac{1}{n}(M_1 + M_2 + \ldots + M_n)$$

Let μ_0 = m. s. e. of the arithmetic mean

μ = m. s. e. of each observed value

Then
$$\mu_0^2 = \frac{1}{n^2}(\mu^2 + \mu^2 + \ldots \text{ to } n \text{ terms})$$

or
$$\mu_0 = \frac{\mu}{\sqrt{n}}$$

That is, *the m. s. e. of the arithmetic mean of n observations is* $\frac{1}{\sqrt{n}}$ *part of that of a single observation.*

On the Comparison of the Accuracy of Different Series of Observations.

20. **The Mean-Square Error.**—We have seen in Art. 16 that the m. s. e. μ affords a test of the relative accuracy of different series of observations. This test was suggested by the fundamental formula of the law of error, and is naturally the first that would be taken for that purpose.

The value of μ^2 is the mean of the sum of the squares of the errors in a series between the extreme limits of error, and since the probability of an error is the number of cases favorable to its occurrence divided by the total number of cases, μ^2 is given by the expression

$$\int_{-a}^{+a} \Delta^2 \varphi(\Delta)\, d\Delta$$

where $+a$ and $-a$ are the limits of error.

Hence if the number of errors n is a very large number a close approximation to the value of μ^2 will be given by

$$\mu^2 = \frac{\Delta_1^2}{n} + \frac{\Delta_2^2}{n} + \ldots + \frac{\Delta_n^2}{n}$$
$$= \frac{[\Delta^2]}{n}.$$

The difference in precision of these two values of μ^2 will be pointed out later. (See Art. 23.)

21. **The Probable Error.**—A second method of determining the relative precision of different series of observations is by comparing errors which occupy the same relative position in the different series when the errors are arranged in order of magnitude. The errors which occupy the middle places in each series are, for greater convenience, the ones chosen.

Let the errors in a series, arranged in order of magnitude, be

$$\pm 2a, \ldots \pm r, \ldots o,$$

each error being written as many times as it occurs; then we give to that error r which occupies the middle place, and which has as many errors numerically greater than it as it has errors less than it, the name of *probable error*. If, therefore, n is the total number of errors, the number lying between $+r$ and $-r$ is $\frac{n}{2}$, and the number outside these limits is also $\frac{n}{2}$. In other words, the probability that the error of a single observation in any system will fall between the limits $+r$ and $-r$ is $\frac{1}{2}$, and the probability that it will fall outside these limits is also $\frac{1}{2}$. We have, therefore,

$$\frac{h}{\sqrt{\pi}}\int_{-r}^{+r} e^{-h^2\Delta^2}\, d\Delta = \frac{1}{2}$$

from which to find r.

If we put $h\Delta = t$, and the value $t = \rho$ corresponds to $\Delta = r$, then

$$\frac{2}{\sqrt{\pi}}\int_{o}^{\rho} e^{-t^2}\, dt = \frac{1}{2}$$

Expanding the integral in a series as in (c) Art. 6, we shall find that approximately the resulting equation is satisfied by

$$\rho = 0.47694$$

Now, since

$$hr = \rho = 0.47694 \text{ and } h\mu\sqrt{2} = 1$$

it follows that

$$r = 0.6745\mu$$
$$= \frac{2}{3}\mu \text{ roughly.}$$

Hence to find the probable error we compute first the mean-square error and multiply it by 0.6745.

As a check, the error which occupies the middle place in the series of errors arranged in order of magnitude may be found. It will be nearly equal to the computed value, if the series is of moderate length.

It is to be clearly understood that the term probable error does not mean that that error is more probable than any other, but only that in a future observation the probability of committing an error greater than the probable error is equal to the probability of committing an error less than the probable error. Indeed, of any single error the most probable is zero. Thus the probability of the error zero is to that of the probable error r as

$$\frac{h}{\sqrt{\pi}} : \frac{h}{\sqrt{\pi}} e^{-h^2r^2}$$

or

$$1 : e^{-(0.47694)^2}$$

or

$$1 : 0.8$$

The idea of probable error is due to Bessel (*Berlin. Astron. Jahrb.*, 1818). The name is not a good one, on account of the word probable being used in a sense altogether different from its ordinary signification. It would be better to use the term *critical error*, for example, as suggested by De Morgan, or *median error*, as proposed by Cournot.

22. **The Average Error.**—It naturally occurs, as a third test of the accuracy of different series of observations, to take the mean of all the positive errors and the mean of all the negative errors, and then, since in a large number of

observations there will be nearly the same number of each kind, to take the mean of the two results without regard to sign. This gives what may be termed the *average error.* It is usually denoted by the Greek letter η.

Reasoning as in Art. 20, we have approximately

$$\eta = \frac{[\Delta}{n}$$

where $[\Delta$ is the arithmetic sum of the errors.

An expression for η in terms of the mean-square error μ may be found as follows. The number of errors between Δ and $\Delta + d\Delta$ is

$$n\,\varphi(\Delta)\,d\Delta$$

and the sum of the positive errors in the series is

$$n\int_0^{\infty} \Delta\,\varphi(\Delta)\,d\Delta$$

The sum of the negative errors being the same, the sum of all the errors is

$$2n\int_0^{+\infty} \Delta\,\varphi(\Delta)\,d\Delta$$

Hence

$$\eta = 2\int_0^{+\infty} \Delta\,\varphi(\Delta)\,d\Delta$$

$$= \frac{2h}{\sqrt{\pi}}\int_0^{+\infty} \Delta e^{-h^2\Delta^2}\,d\Delta$$

$$= \frac{1}{h\sqrt{\pi}} = \mu\sqrt{\frac{2}{\pi}}$$

the relation required.

The average error may, as stated above, be directly used as a test of the relative accuracy of different series of observations. Indeed, I think it should be more used for this purpose than it is. The general custom is, however, to employ it as a stepping-stone to find the mean-square and probable errors. This can be done, for the reason that it is more easy to compute $[\Delta$ than $[\Delta^2]$.

The formulas for μ and r computed in this way are as follows. From the last equation preceding

$$\mu = \sqrt{\frac{\pi}{2}}\,\eta$$

$$= 1.2533\,\frac{[\varDelta}{n}$$

and from Art. 21

$$r = 0.6745\,\mu$$

$$= 0.8453\,\frac{[\varDelta}{n}$$

The relations connecting μ, r, and η are easily remembered in the following form:

$$\mu\sqrt{2} = \frac{r}{\rho} = \sqrt{\pi}\,\eta$$

These relations may also be conveniently arranged in tabular form:

	μ	r	η
$\mu =$	1.0000	1.4826	1.2533
$r =$	0.6745	1.0000	0.8453
$\eta =$	0.7979	1.1829	1.0000

23. We have seen that the m. s. e. μ may be computed from the sum of the squares of the errors and also from the sum of the errors without regard to sign. In the derivation of each formula certain approximations have been made. The question then arises which of the two methods will give the more reliable result. This will be shown by the ranges in the two determinations of the value of μ.

(a) We proceed to find first the m. s. e. in the determination of μ^2 by taking the approximate formula

$$\frac{[\varDelta^2]}{n}\ (= \mu_1^2 \text{ suppose})$$

instead of the rigid formula

$$\int_{-\infty}^{+\infty} \Delta^2 \varphi(\Delta)\, d\Delta$$

If we put

$$\lambda = \mu_1^2 - \mu^2$$

then, since μ^2 is the exact value, λ will be the error of μ^2 computed from the errors $\Delta_1, \Delta_2 \ldots$ according to the formula $\frac{[\Delta^2]}{n}$. Squaring, we have

$$\lambda^2 = \mu_1^4 - 2\mu_1^2\mu^2 + \mu^4$$
$$= \frac{[\Delta^4]}{n^2} + \frac{2}{n^2}(\Delta_1^2\Delta_2^2 + \Delta_1^2\Delta_3^2 + \ldots) - 2\mu^2\frac{[\Delta^2]}{n} + \mu^4$$

Now, letting the errors Δ assume all possible values, the average value of the fourth powers is (see Art. 6)

$$\frac{2h}{\sqrt{\pi}}\int_0^\infty \Delta^4 e^{-h^2\Delta^2} d\Delta = \frac{3}{4h^4}$$
$$= 3\mu^4$$

The number of the products $\Delta_1^2\Delta_2^2$, $\Delta_1^2\Delta_3^2$, ... being the number of combinations of n things, two at a time, is $\frac{n(n-1)}{2}$ and the average of the values is

$$\frac{n(n-1)}{2}\left\{\frac{2h}{\sqrt{\pi}}\int_0^\infty \Delta^2\, e^{-h^2\Delta^2}\, d\Delta\right\}^2$$

that is (Art. 20),

$$\frac{n(n-1)}{2}\,\mu^4$$

The average value of $\mu^2\frac{[\Delta^2]}{n}$ is μ^4. Hence finally

$$\lambda^2 = \frac{3\mu^4}{n} + \frac{2}{n^2}\,\frac{n(n-1)}{2}\,\mu^4 - 2\mu^4 + \mu^4$$
$$= \frac{2\mu^4}{n}$$

and

$$\mu_1^2 = \mu^2 \pm \mu^2 \sqrt{\frac{2}{n}}$$

or

$$\mu_1 = \mu\left(1 \pm \sqrt{\frac{2}{n}}\right)^{\frac{1}{2}}$$

$$= \mu\left(1 \pm \frac{1}{\sqrt{2n}}\right) \text{ when } n \text{ is very large.}$$

(b). In the second place, for the average error η we proceed in precisely the same way. We have

$$\eta = 2\int_0^\infty \Delta\varphi(\Delta)\,d\Delta$$

$$\eta_1 = \frac{[\Delta}{n}$$

Let $\lambda_1 = \eta_1 - \eta$

Then $\lambda_1^2 = \eta_1^2 - 2\eta_1\eta + \eta^2$

$$= \frac{[\Delta^2]}{n^2} + \frac{2}{n^2}(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \ldots) - 2\eta\frac{[\Delta}{n} + \eta^2$$

$$= \frac{\mu^2}{n} + \frac{2}{n^2}\,\frac{n(n-1)}{2}\,\frac{2\mu^2}{\pi} - \eta^2$$

$$= \frac{\mu^2}{n}\left(1 - \frac{2}{\pi}\right)$$

which gives the error in η.

Also, since

$$\mu = \sqrt{\frac{\pi}{2}}\,\eta$$

the error in μ is

$$\frac{\mu}{\sqrt{n}}\sqrt{\frac{\pi}{2}}\sqrt{1 - \frac{2}{\pi}}; \text{ that is, } \mu\sqrt{\frac{\pi - 2}{2n}}$$

Hence by this method of computing μ the value of μ is contained between the mean limits

$$\mu\left(1 \pm \sqrt{\frac{\pi - 2}{2n}}\right)$$

Now, since $\pi - 2 > 1$, the limits in the latter case are the larger, and we therefore conclude that the former method of computing μ is the better of the two.

24. From the equation

$$\mu = \eta \sqrt{\frac{\pi}{2}}$$

we may derive a test of the validity of the law of error, and a rather curious one. For μ and η may be determined from measurements, and if the experimental values found satisfy the equation

$$\frac{\mu^2}{\eta^2} = \frac{\pi}{2}$$

we must conclude that the theory is correct. This may be classed as an additional *à posteriori* proof to that given in Art. 51.

25. Whether we should use the m. s. e. or the p. e. in stating the precision is largely a matter of taste. Gauss says: "The so-called probable error, since it depends on hypothesis, I, for my part, would like to see altogether banished; it may, however, be computed from the mean by multiplying by 0.6744897." On the other hand, the International Committee of Weights and Measures decided in favor of the probable error: "It has been thought best in this work that the measure of precision of the values obtained should always be referred to the probable error computed from Gauss' formula, and not to the mean error." (*Procès Verbaux*, 1879, p. 77.)

In the United States, in the Naval Observatory, the Coast Survey, the Engineer Corps, and the principal observatories, the p. e. is used altogether. So, too, in Great Britain, in the Greenwich Observatory, the Ordnance Survey, etc. In the G. T. Survey of India the m. s. e. is used, for the reason given by Gauss above. Among German geodeticians and astronomers the m. s. e. is very generally employed.

In this book the m. s. e. will be used in the text, and the m. s. e. and p. e. in the examples indifferently.

The Probability Curve.

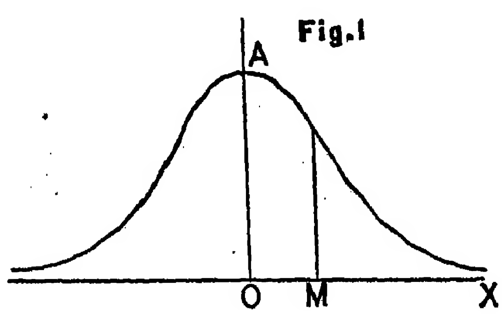
26. The principles laid down in the preceding articles may be illustrated geometrically as follows:

We have seen that in a series of observations the probability that an error will lie between the values x and $x+dx$ is given by the expression

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx$$

Now, if O is the origin of co-ordinates, and a series of errors, x, are represented by the distances from O along the axis of abscissas OX, positive errors being taken to the right of O and negative errors to the left, then the probability, in a future observation, of an error falling between x and $x+dx$ will be represented by the rectangle whose height is $\frac{h}{\sqrt{\pi}} e^{-h^2x^2}$ and width dx, or, more strictly, by the ratio of this rectangle to the sum of all such rectangles between the extreme limits of error. This sum we have for convenience already denoted by unity.

Fig. 1



Hence for a series of observations whose quality is known, by giving to x all values from $+\infty$ to $-\infty$ and drawing the corresponding ordinates, we shall have a continuous curve whose equation may be written

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$$

This curve is called the probability curve.

27. **To Trace the Form of the Curve.**—Since x enters to the second power, and y to the first power, the curve is symmetrical with respect to the axis of y, and the form of the equation shows that it lies altogether on one

side of the axis of x. Also, when $x = 0$, $\frac{dy}{dx} = 0$; that is, the tangent at the vertex is parallel to the axis of x.

As x increases from 0 the values of y continually decrease. When $x = \pm \infty$, then

$$y = 0 \text{ and } \frac{dy}{dx} = 0$$

showing that the axis of x is an asymptote.

Again, since

$$\frac{d^2y}{dx^2} = \frac{2h^3}{\sqrt{\pi}} e^{-h^2x^2}(2h^2x^2 - 1)$$

there is a point of inflection when

$$x = \frac{1}{h\sqrt{2}} = \mu$$

and the m. s. e. is therefore the abscissa of the point of inflection. Also, when $x = 0$, $\frac{d^2y}{dx^2}$ is negative, showing that the ordinate at the vertex is the maximum ordinate. Hence the curve is of the form indicated in Fig. 1, OA representing the maximum ordinate and OM the m. s. e.

The values of h, that is, of $\frac{1}{\mu\sqrt{2}}$, being different for different series of observations, the form of the curve will change for each series, and the curve may be plotted to scale from values of y corresponding to assumed values of x.

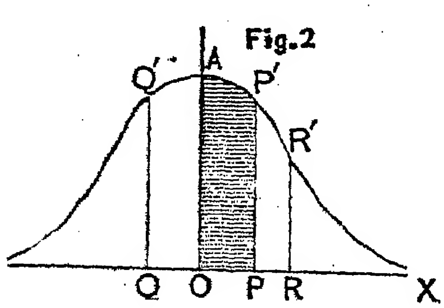
In plotting the curve, since the maximum ordinate at the vertex $\frac{h}{\sqrt{\pi}}$ enters as a factor into the values of each of the other ordinates, its value may be arbitrarily assumed. We may therefore adopt a scale for plotting the ordinates different from the scale by which the abscissas are plotted, in order to show the curve more clearly.

The form of the curve is in accordance with the principles already laid down in deducing the law of error, and

could have been derived from them directly. Thus that small errors are more probable than large is indicated by the element rectangle areas being greater for values of x near zero than for values more distant; that very large errors have a very small probability is indicated by the asymptotic form of the curve, and that positive and negative errors are equally probable is indicated by its symmetrical form with respect to the axis of y.

28. The area of the curve of probability being the sum of the rectangles $\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx$, for values of x extending from $+\infty$ to $-\infty$ we have denoted by unity. If, then, we represent by the area of this curve the total number of errors that occur in a series of observations, it follows from the definition of probability that the area included between certain assigned limits would represent the number of errors to be expected in the series between the values of those limits.

Fig. 2



Thus if O is the origin, the area to the right of OA would represent the number of positive errors and the area to the left of OA the number of negative errors. The area $OPP'A$ would represent the number of positive errors less than OP, the area $PRR'P'$ the number that lie between OP and OR.

If the area $AOPP'$ is equal to one-half the total area AOX, then the number of positive errors less than OP would be equal to the number greater than OP. Hence OP would represent the probable error. If OQ be taken equal to OP, the area $PQQ'P'$ would represent the number of errors numerically less than the probable error.

29. The average error η may be illustrated as follows: If x_1 is the abscissa of the centre of gravity of any curve referred to the axes of X and Y, then

$$x_1 = \frac{\int yx\,dx}{\int y\,dx}$$

Applying this to the curve of probability and calling the whole area unity, we have

$$x_1 = \int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} e^{-h^2x^2}\, x\, dx$$

$$= \frac{1}{h\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} t\, dt \text{ if } hx = t$$

$$= \frac{1}{h\sqrt{\pi}}$$

$$= \eta, \text{ the average error.}$$

The Law of Error applied to an Actual Series of Observations.

We here bridge over the gulf between the ideal series from which we have derived the law of error and the actual series with which we have to deal in practical work, and which can only be expected to come partially within the range of the law constructed for the ideal.

30. **Effect of Extending the Limits of Error to $\pm\infty$.**—The expression $\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta$ gives the value of the probability of an error between Δ and $\Delta + d\Delta$ in an ideal series of observations where the values are continuous between limits infinitely great. In all actual series the possible error is included within certain finite limits, and the probability of the occurrence of an error beyond those limits is zero. Practically, however, the extension of the limits of error to $\pm\infty$ can make no appreciable difference in either case, as the function $\varphi(\Delta)$ decreases so rapidly that we can regard it as infinitesimal for large values of Δ; in other words, the greater number of errors is in the neighborhood of zero, and therefore the most important part is the part covered by both. This has been illustrated geometrically in the discussion of the probability curve, and will now be developed from another point of view.

We have for the probability of the occurrence of an error not greater than a, in a series of observations,

$$\frac{2h}{\sqrt{\pi}}\int_0^a e^{-h^2\Delta^2}\,d\Delta$$

This may be put in the form ($t = h\Delta$)

$$\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt$$

and is usually denoted by the symbol $\Theta(t)$.

The method of evaluating $\Theta(t)$ has been explained in (c) Art. 6. In Table I. will be found the values of the function $\Theta(t)$ corresponding to the argument $\frac{ah}{\rho}$, that is, $\frac{a}{r}$. The reason for arranging the table in this way is that it is more convenient to compute $\frac{a}{r}$ than $\frac{a}{r}\rho$.

The probability that an error exceeds a certain error a is $1 - \Theta(t)$, and may be found from Table I. by deducting the tabular value from unity. Thus we have the probability that a is greater than r is 0.5, than $2r$ is 0.1773, than $3r$ is 0.0430, than $4r$ is 0.0070, than $5r$ is 0.0007, than $6r$ is 0.0001.

Hence in 10,000 observations we should expect only one error greater than $6r$, in 1,000 only one greater than $5r$, in 100 only one greater than $4r$, and in 25 only one greater than $3r$. If in any set of observations we found results much at variance with these we could assume that they arose from some unusual cause, and should, therefore, be specially examined. As in practice the number of observations in any case is usually under 100, we are eminently safe in taking the maximum error at about $5r$ or 3μ.

We are now in a position to estimate the change introduced by replacing c as found from

$$c\int_{-a}^{+a} \epsilon^{-h^2\Delta^2}\,d\Delta = 1$$

by $\frac{h}{\sqrt{\pi}}$ as found from

$$c \int_{-\infty}^{+\infty} e^{-h^2\Delta^2}\, d\Delta = 1$$

in the investigation of the law of error.

These equations may be written

$$2c \int_0^a e^{-h^2\Delta^2}\, d\Delta = 1$$

$$\frac{2h}{\sqrt{\pi}}\left(\int_a^\infty e^{-h^2\Delta^2}\, d\Delta + \int_0^a e^{-h^2\Delta^2}\, d\Delta\right) = 1$$

Hence

$$c = \frac{h}{\sqrt{\pi}}\left(1 + \frac{2h}{\sqrt{\pi}} \int_a^\infty e^{-h^2\Delta^2}\, d\Delta\right) \text{ approximately,}$$

and taking $a = 5r$, we have, from Table I.,

$$c = \frac{h}{\sqrt{\pi}}\,(2 - 0.9993)$$

$$= 1.001\,\frac{h}{\sqrt{\pi}}$$

Hence the difference being less than $\frac{1}{1000}$ of the quantity sought, the approximate value of c found by extending the limits $\pm a$ to $\pm \infty$ may be considered satisfactory.

31. **Various Laws of Error.**—We have taken the arithmetic mean of a series of observed values of a quantity made under like conditions as the most plausible value of the quantity. The supposition of each observed quantity being subject to the same law of error leads to the mean as the most probable value. "The method of least squares is, in fact, a method of means, but with some peculiar characters. The method proceeds upon this supposition, that all errors are not equally probable, but that small errors are more probable than large ones." *

Now, in an ordinary series we assume a good deal when we take each observation of the series as subject to the same

* Whewell, *History of the Inductive Sciences*, vol. ii.

special law of error—the exponential law. We can certainly conceive of laws different from this one. It is more probable that each set of observations has its own law depending on instrument, observer, and conditions. If we could go back to the sources of error we could find this law in each case. Let us follow out this idea in a few simple cases and see to what it leads:

32. First take the case where all errors are equally probable—that is, where Δ can with the same probability assume all values between $+a$ and $-a$, the extreme limits of error.

Since a is the maximum error,

$$\varphi(\Delta)\int_{-a}^{+a} d\Delta = 1, \ \varphi(\Delta) \text{ being constant,}$$

and therefore

$$\varphi(\Delta) = \frac{1}{2a}$$

the law of error.

For the m. s. e. μ we have

$$\mu^2 = \int_{-a}^{+a} \frac{\Delta^2}{2a}\, d\Delta$$

$$= \frac{a^2}{3}$$

Also

$$\eta = \int_{-a}^{+a} \varphi(\Delta)\, \Delta\, d\Delta$$

$$= \int_{-a}^{+a} \frac{\Delta\, d\Delta}{2a}$$

$$= \frac{a}{2}$$

The p. e. is found from the relation

$$\int_{-r}^{+r} \varphi(\Delta)\, d\Delta = \frac{1}{2}$$

or

$$\int_{-r}^{+r} \frac{d\Delta}{2a} = \frac{1}{2}$$

and $$r = \frac{a}{2}$$

that is, the p. e. is half the max. error.

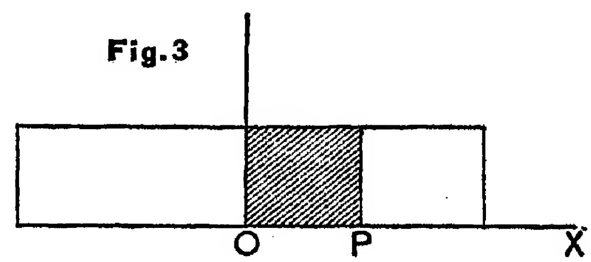
This is also evident geometrically, for the curve of error

$$y = \text{constant}$$

is a straight line parallel to the axis of abscissas.

Hence by definition we find the p. e., by bisecting the area, to be $\frac{a}{2}$. The p. e. would be represented in the figure by OP.

Fig. 3



33. Next consider an error to arise from two independent sources, x, y, each of which can with the same probability assume all values between $+a$ and $-a$, so that for the total error Δ we have

$$\Delta = x + y$$

To find $\varphi(\Delta)$, the law of error: If Δ and $\Delta + d\Delta$ denote two consecutive errors, then since x and y have values between $+a$ and $-a$ with interval $d\Delta$, each of the quantities x, y has $\frac{2a}{\Delta}$ possible values, and the whole number of possible causes of error is $\frac{2a}{d\Delta} \times \frac{2a}{d\Delta}$ or $\frac{4a^2}{(d\Delta)^2}$. The causes which are favorable for an error of the magnitude $+\Delta$ answer to all possible values of x, y which together give $+\Delta$; namely,

$$\begin{array}{llllll} \text{for } x, & \Delta - a, & \Delta - a + d\Delta, & \Delta - a + 2d\Delta, \ldots & a - d\Delta, & a \\ \text{for } y, & a, & a - d\Delta, & a - 2d\Delta, \ldots & \Delta - a + d\Delta, & \Delta - a \end{array}$$

and whose number is therefore $\frac{2a - \Delta}{d\Delta}$.

In the same way we find the number of causes favorable for the error of the magnitude $-\Delta$ to be $\frac{2a + \Delta}{d\Delta}$.

Hence the probability $\varphi(\varDelta)\,d\varDelta$ of an error between and $\varDelta + d\varDelta$ is given by

$$\varphi(\varDelta)\,d\varDelta = \frac{2a \pm \varDelta}{d\varDelta} \div \frac{4a^2}{(d\varDelta)^2}$$
$$= \frac{2a \pm \varDelta}{4a^2}\,d\varDelta$$

that is,

$$\varphi(\varDelta) = \frac{2a + \varDelta}{4a^2} \text{ when } \varDelta \text{ lies between } -2a \text{ and } 0$$

$$\varphi(\varDelta) = \frac{2a - \varDelta}{4a^2} \text{ when } \varDelta \text{ lies between } 0 \text{ and } +2a.$$

For the m. s. e. we have

$$\mu^2 = \int_0^{2a} \varDelta^2\,\frac{2a - \varDelta}{4a^2}\,d\varDelta + \int_{-2a}^{0} \varDelta^2\,\frac{2a + \varDelta}{4a^2}\,d\varDelta$$
$$= \frac{2}{3}\,a^2$$

For the p. e.,

$$\int_0^{r} \frac{2a - \varDelta}{4a^2}\,d\varDelta + \int_{-r}^{0} \frac{2a + \varDelta}{4a^2}\,d\varDelta = \frac{1}{2}$$
$$\therefore r = (2 - \sqrt{2})\,a$$

Geometrically, the equations

$$y = 2a - x$$
$$y = 2a + x$$

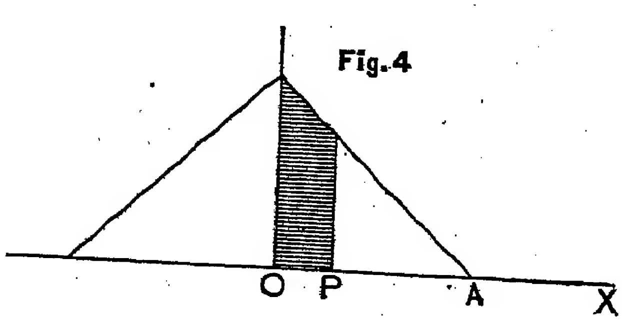


Fig. 4

represent two straight lines which cut the axes of x and y at an angle of 45°, and therefore the curve of probability is as in the figure. If in the figure $OA = 2a$, then OP represents the p. e.

34. From the preceding we may derive an important practical point. In an ordinary seven-place log. table the seventh place is never in error by more than 0.5. Hence,

this being the maximum error, the p. e. of a log. as given in the tables is 0.25 in the seventh place. The interpolated value at the greatest distance from a tabular value is the mean between two tabular values. Its p. e., from Art. 33, is

$$(2 - \sqrt{2}) \times 0.25 = 0.15$$

Hence the p. e. of the log. of a number may be taken 0.2 in the seventh decimal place. The p. e. of the number corresponding to this log. is (Ex. 1, Art. 7)

$$\frac{2}{10^8 \times \text{mod.}} = \frac{1}{22 \times 10^6} \text{ approx.}$$

Suppose now that we are computing a chain of triangles starting from a measured base. The p. e. of the base may be taken as $\frac{1}{1,000,000}$ of its length. Hence the error arising from this source is 22 times that to be expected from the log. tables. Again, the triangulation will be most exact, and therefore the test most severe, when the angles of each triangle are equal to 60°. Now, the change in log sin 60° corresponding to a change of 1″ is 12.2 in units of the seventh decimal place. And in a primary triangulation an angle may, with the instruments now in use, be measured with a p. e. of 0″.25. Hence

$$\begin{aligned}\text{p. e. of log sin } 60 &= 12.2 \times 0.25 \\ &= 3.0 \text{ in units of the seventh decimal place,}\end{aligned}$$

which p. e. is 15 times greater than that arising from the log. tables.

For the solution of triangles, therefore, we conclude that, with our present means of measurement, seven-place tables are sufficient. The common practice is to carry out to eight places to give greater accuracy in the seventh place, and then drop the eighth place in stating the final result. (See Struve, *Arc du Méridien*, vol. i. p. 94.)

35. If an error Δ arises from three independent sources of the same kind, each of which can with the same probability assume all values between $+a$ and $-a$, then, the

maximum error being $3a$, we have, from similar reasoning to that employed in Art. 33,

$$\varphi(\Delta) = \frac{(3a - \Delta)^2}{16a^3} \text{ when } \Delta \text{ is between } + 3a \text{ and } + a$$

$$\varphi(\Delta) = \frac{3a^2 - \Delta^2}{8a^3} \text{ when } \Delta \text{ is between } + a \text{ and } - a$$

$$\varphi(\Delta) = \frac{(3a + \Delta)^2}{16a^3} \text{ when } \Delta \text{ is between } - a \text{ and } - 3a$$

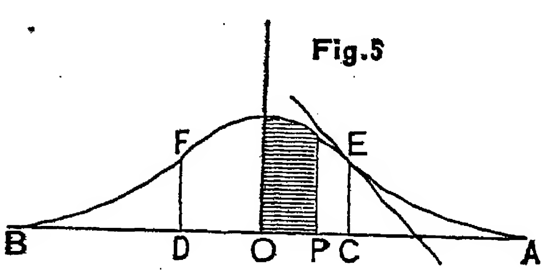
Also

$$\mu^2 = 2\int_0^a \frac{3a^2 - \Delta^2}{8a^3} \Delta^2 d\Delta + 2\int_a^{3a} \frac{(3a - \Delta)^2}{16a^3} \Delta^2 d\Delta$$
$$= a^2$$

and

$$\int_{-r}^{+r} \frac{3a^2 - \Delta^2}{8a^3} d\Delta = \frac{1}{2}$$
$$\therefore r = 0.71a$$

The curve of probability consists of three parts, as in the figure:



$OA = OB = 3a \qquad OC = OD = a$

There are common tangents to the two branches at E and F, and the curve touches the axis of X at A and B. The p. e. is represented by OP.

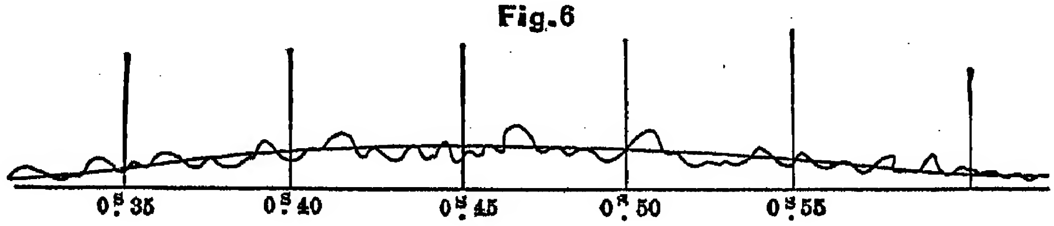
36. A consideration of the results obtained in Arts. 31–35 will show that the more numerous the sources of error assumed the nearer we approach the results obtained from the Gaussian law of error. Thus for

one source	$\frac{r}{\mu} = 0.87$	$\frac{r}{\eta} = 1.00$
two sources	$\frac{r}{\mu} = 0.72$	$\frac{r}{\eta} = 0.88$
three sources	$\frac{r}{\mu} = 0.71$	$\frac{r}{\eta} = 0.87$
Gaussian law	$\frac{r}{\mu} = 0.67$	$\frac{r}{\eta} = 0.85$

The forms of the curves of probability show the same approach to coincidence. Starting with a straight line as the curve for a single source of error, we approach quite closely to the Gaussian probability curve, even with so small a number of sources of error as three. Hence we should expect that, starting from the postulate of an error being derived from the combined influence of a very large number of independent sources of error, we should arrive at the Gaussian law of error. A complete demonstration of this by Bessel, to whom the idea itself is due, will be found in *Astron. Nachr.*, Nos. 358, 359. The elementary proof given in Art. 33 for the simple case of two sources of error is due to Zachariæ.

37. **Experimental Proof of the Law of Error.**— The same point was brought out experimentally in a series of researches by Prof. C. S. Peirce, of the U. S. Coast Survey.* He employed a young man, who had had no previous experience whatever in observing, to answer a signal consisting of a sharp sound like a rap, the answer being made upon a telegraph operator's key nicely adjusted. Five hundred observations were made on each of twenty-four days. The results for the first and last days are plotted below. In the

Fig. 6



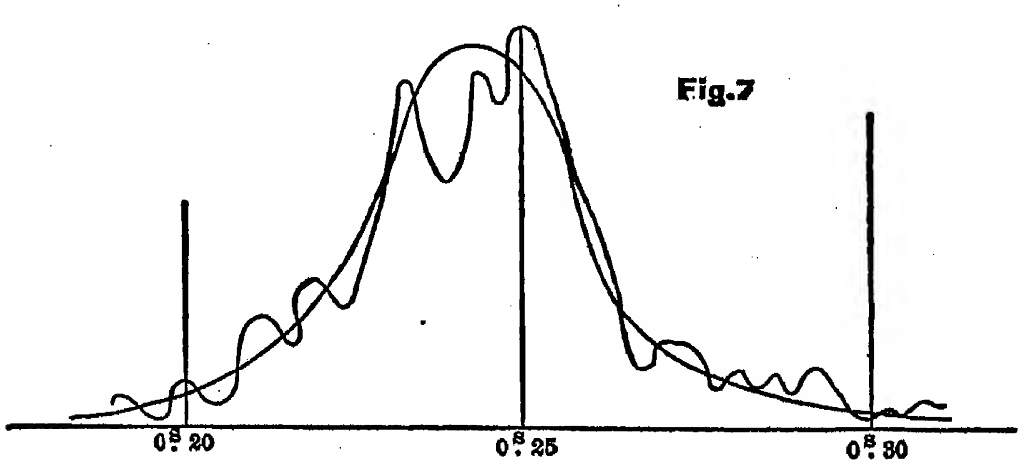


Fig. 7

* *Coast Survey Report*, 1870, appendix 21.

figures the abscissas represent the interval of time between the signal and the answer, the ordinates the number of observations. The curve is a mean curve for every day, drawn by eye so as to eliminate irregularities entirely. After the first two or three days the curve differed very little from that derived from the theory of least squares. On the first day, when the observer was entirely inexperienced, the observations scattered to such an extent that the curve had to be drawn on a different scale from that of the other days.

38. **General Conclusions.**—On the whole, though we cannot say that the formula $\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$ will truly represent the law of error in any given series of observations, we can say that it is a close approximation.

When in a series of observations we have exhausted all of our resources in finding the corrections, and have applied them to the measured values, the residuum of error may fairly be supposed to have arisen from many sources; and we conclude from the foregoing investigations that, of any one single law, the best to which we can consider the residual errors subject, and the best to be applied to a set of observations not yet made, is the exponential law of error.

The general theorem of Art. 17 may therefore be applied to a limited series and be written: *If the observed values of a quantity are of different quality, the most probable value is found by dividing each residual error by the m. s. e. and making the sum of the squares of the quotients a minimum; if of the same quality, the most probable value is the arithmetic mean of the observed values.*

If a set of observations shows a marked divergence from this law a rigid examination will reveal the necessity, in general, of applying some hitherto unknown correction. Thus in the earlier differential comparisons of the compensating base-apparatus of the United States Lake Survey with the standard bar packed in ice, the observed differences did not follow the law of error, as it was fair to suppose that they should, the bars being compensating. There was in-

stead a regular daily cycle: some one source of error so far exceeded the others that it overshadowed them. A study of the results was made, and the law of daily change discovered, which gave a means of applying a further correction. The work done later, after taking account of this new correction, showed nothing unusual.

Classification of Observations.

39. For purposes of reduction observations may be divided into two classes—those which are independent, being subject to no conditions except those fixed by the observations themselves, and those which are subject to certain conditions outside of the observations, as well as to the conditions fixed by the observations. In the former class, before the observations are made, any one assumed set of values is as likely as any other; in the latter no set of values can be assumed to satisfy approximately the observation equations which does not exactly satisfy the *à priori* conditions.

For example, suppose that at a station O the angles AOB, AOC are measured. If the measures of each angle are independent of those of the other, the angles are found directly.

The angle BOC could be determined from the relation

$$AOC = AOB + BOC$$

The unknown in this case may be said to be observed indirectly, and therefore independent observations may be classed as *direct* and *indirect*. The former class is a special class of the latter.

But if the angle BOC is observed directly as well as AOB, AOC, then these angles are no longer independent, but are subject to the condition that when adjusted

$$AOC = AOB + BOC$$

and no set of values can be assumed as possible which does not exactly satisfy this condition.

The observations in this case are said to be *conditioned.* Though we have, therefore, strictly speaking, only two classes of observations, we shall, for simplicity, divide the first into two and consider in order the adjustment of

(1) Direct observations of one unknown.

(2) Indirect observations of several independent unknowns.

(3) Condition observations.

CHAPTER III.

ON THE ADJUSTMENT OF DIRECT OBSERVATIONS OF ONE UNKNOWN QUANTITY.

IN the application of the ideal formulas of Chapter II. to an actual series of observations we shall begin with a single quantity which has been directly observed. We shall consider two cases—first, when all of the observed values are of equal quality, and, next, when they are not all of equal quality.

A. *Observed Values of Equal Quality.*

40. **The Most Probable Value; the Arithmetic Mean.**—We have seen that in a series of directly observed values $M_1, M_2, \ldots M_n$ of equal quality the most probable value V of the observed quantity is found by taking the arithmetic mean of these values; that is,

$$V = \frac{[M]}{n} \qquad (1)$$

It has also been shown that the same result will follow by making the sum of the squares of the residual errors a minimum. Thus the observations give the equations,

$$\begin{aligned} V - M_1 &= v_1 \\ V - M_2 &= v_2 \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ V - M_n &= v_n \end{aligned} \qquad (2)$$

and V is to be found from

$$v_1^2 + v_2^2 + \ldots + v_n^2 = \text{a min.} \qquad (3)$$

that is, from

$$(V - M_1)^2 + (V - M_2)^2 + \ldots + (V - M_n)^2 = \text{a min.} \qquad (4)$$

By differentiation of (4)

$$(V-M_1)+(V-M_2)+\ldots+(V-M_n)=0 \qquad (5)$$

and

$$V=\frac{[M]}{n} \qquad (6)$$

In practice it would evidently be simpler to find the value of the unknown by taking the arithmetic mean of the observed values directly rather than to form the observation equations and find it by making the sum of the squares of the residuals a minimum.

It is useful to notice, for purposes of checking, that Eq. (5) may be written

$$[v]=0 \qquad (7)$$

41. As the observed values M are often numerically large and not widely different, the arithmetical work of finding the mean may be shortened as follows:

A cursory examination of the observations will show about what the mean value V must be. Let X' denote this approximate value of V, which may conveniently be taken some round number. Subtract X' from each of the observed values $M_1, M_2, \ldots M_n$ in succession, and call the differences $l_1, l_2, \ldots l_n$ respectively. Then

$$\begin{aligned} M_1 - X' &= l_1 \\ M_2 - X' &= l_2 \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ M_n - X' &= l_n \end{aligned} \qquad (1)$$

By addition,

$$\begin{aligned} [M] - nX' &= [l] \\ \therefore V &= \frac{[M]}{n} \\ &= X' + \frac{[l]}{n} \\ &= X' + x' \text{ suppose.} \end{aligned} \qquad (2)$$

Hence all that we have to do is to take the mean x' of the small quantities $l_1, l_2, \ldots l_n$, and add the assumed value X' to the result.

Ex. The measured values of an angle are

$$177^\circ \quad 21' \quad 5''.80$$
$$177^\circ \quad 21' \quad 7''.35$$
$$177^\circ \quad 21' \quad 4''.28$$

Find the mean.

It is sufficient to find the mean of the seconds and carry in the degrees and minutes unchanged.

42. Control of the Arithmetic Mean.—In least squares, as in all computations, it is important to have a check or control of the numerical work. This is specially desirable when a computation takes several weeks, or it may be months, to complete it. In long computations it is better for two computers to work together, using different methods whenever possible, and to compare results at intervals. But even this is not an absolute safeguard against mistakes, as it sometimes happens that both make the same slip, as, for example, writing $+$ for $-$, or *vice versa*. Hence, even if the computation is made in duplicate, it is advisable to carry through an independent check which may be referred to occasionally. In computations not duplicated a control is essential.

A control of the accuracy of the arithmetic mean of a set of observed values of the same quantity is afforded by the relation

$$[v] = 0$$

that is, that the sum of the positive residuals should be equal to the sum of the negative residuals.

If, however, in finding the arithmetic mean, the sum $[M]$ of the observed quantities was not exactly divisible by their number n, the sums of the positive and negative residuals would not be equal, but the amount of the discrepancy could easily be estimated and allowed for. For if the value of the mean taken were too large by a certain amount, the positive residuals would each be too large, and the negative residuals also too large, by that amount. Hence the discrepancy to be expected would be n times the amount that the approximate quotient taken as the mean differed from the exact quotient.

Ex. In the telegraphic determination of the difference of longitude between St. Paul and Duluth, Minn., June 15, 1871, the following were the corrections found for chronometer Bond No. 176 at 15h. 51m. sidereal time from the observations of 21 time stars. (*Report Chief of Engineers U. S. A.*, 1871.)

M	v		vv
s.	s.		
− 8.78	+ 0.04		0.0016
.76	+ .02		4
.85	+ .11		121
.78	+ .04	s.	16
.51		− 0.23	529
.64		− .10	100
.68		− .06	36
.63		− .11	121
.58		− .16	256
.80	+ .06		36
.75	+ .01		1
.78	+ .04		16
.96	+ .22		484
.64		− .10	100
.65		− .09	81
.83	+ .09		81
.70		− .04	16
.64		− 0.10	100
.79	+ .05		25
.90	+ .16		256
− 8.93	+ 0.19		0.0361
Mean − 8.74	+ 1.03	− 0.99	$[vv] = 0.2756$
	$[v = 2.02$		

Taking the observations as of equal precision, we find the arithmetic mean to be − 8.74. This is the most probable value of the correction.

The residuals v are found by subtracting each observed value from the most probable value according to the relation

$$V - M = v$$

They are written in two columns for convenience in applying the check

$$[v] = 0$$

The exact mean being $-8.74\frac{4}{21}$, the quantity $\frac{4}{21} \times 21 = 4$ should be, as it is, the numerical difference between the + and − residuals. Hence we may consider the mean to be correctly found.

Precision of the Arithmetic Mean.—The degree of confidence to be placed in the most probable value of the unknown is shown by its mean-square or probable error.

43. (a) *Bessel's Formula.*

If we knew the true value T of the unknown, and consequently the true errors $\Delta_1, \Delta_2 \ldots$ we should have, as in Art. 20, for the m. s. e. of an observation,

$$\mu^2 = \frac{[\Delta\Delta]}{n}$$

But we have only the most probable value V and the residual errors $v_1, v_2, \ldots v_n$ instead of the true values $T, \Delta_1, \Delta_2 \ldots \Delta_n$. Now,

$$\begin{aligned} V - v_1 &= M_1 = T - \Delta_1 \\ V - v_2 &= M_2 = T - \Delta_2 \\ &\cdot \quad \cdot \quad \cdot \\ V - v_n &= M_n = T - \Delta_n \end{aligned} \qquad (1)$$

By addition, remembering that $[v] = 0$,

$$nV = nT - [\Delta] \qquad (2)$$

Substitute for V in equations (1) and

$$\begin{aligned} nv_1 &= (n-1)\Delta_1 - \Delta_2 - \ldots \\ nv_2 &= \quad -\Delta_1 + (n-1)\Delta_2 - \ldots \\ &\cdot \quad \cdot \quad \cdot \end{aligned}$$

Squaring,

$$\begin{aligned} n^2v_1^2 &= (n-1)^2\Delta_1^2 + \Delta_2^2 + \ldots - 2(n-1)\Delta_1\Delta_2 - \ldots \\ n^2v_2^2 &= \Delta_1^2 + (n-1)^2\Delta_2^2 + \ldots - 2(n-1)\Delta_1\Delta_2 - \ldots \\ &\cdot \quad \cdot \quad \cdot \end{aligned}$$

By addition, assuming that the double products destroy each other, positive and negative errors being equally probable,

$$n^2[vv] = \{(n-1)^2 + (n-1)\}[\Delta\Delta]$$

$$\therefore [vv] = \frac{n-1}{n}[\Delta\Delta]$$

$$= (n-1)\mu^2$$

and

$$\mu^2 = \frac{[vv]}{n-1} \qquad (3)$$

which gives the m. s. e. of an observation.

Now, from Art. 19,

$$\mu_o = \frac{\mu}{\sqrt{n}}$$

$$\therefore \mu_o = \sqrt{\frac{[vv]}{n(n-1)}}$$

which gives the m. s. e. of the arithmetic mean of n observations of equal precision.

The result, $\mu^2 = \frac{[vv]}{n-1}$, might have been inferred *à priori.* For the series of residuals $v_1, v_2 \ldots$ found from the arithmetic mean V of the observed values approximates closely to the true series of errors Δ from which the law of error was derived. Hence we conclude that the formula

$$\mu^2 = \frac{[vv]}{n} \tag{4}$$

would be a close approximation to the m. s. e. of an observation. It is, however, not satisfactory, from the fact that it ought to become indeterminate when $n = 1$, which it does not. For when $n = 1$, $v = 0$, and unless the denominator of (4) is equal to 0, μ would be equal to 0; that is, the first observation would give the true value of the unknown, which is absurd. Hence we should expect the formula to be of the form

$$\mu^2 = \frac{[vv]}{n-1}$$

which becomes of the indeterminate form $\frac{0}{0}$ when $n = 1$.

44. As in Art. 23, we may show from the expansion of $\left(\frac{[vv]}{n-1} - \mu^2\right)^2$ that the square of the m. s. e. of $\mu^2 = \frac{[vv]}{n-1}$ is equal to $\mu^2 \sqrt{\frac{2}{n-1}}$. We have, therefore,

$$\mu^2 = \frac{[vv]}{n-1}\left(1 \pm \sqrt{\frac{2}{n-1}}\right)$$

and when n is very large

$$\mu = \sqrt{\frac{[vv]}{n-1}}\left(1 \pm \frac{1}{\sqrt{2(n-1)}}\right)$$

that is, the mean uncertainty of μ is

$$\pm \frac{\mu}{\sqrt{2(n-1)}}$$

45. From the constant relation existing between the m. s. e. and p. e. given in Art. 22 we have for the p. e. of an observation and of the arithmetic mean of n observations respectively,

$$\begin{aligned} r &= \rho\sqrt{2}\,\mu \\ &= \rho\sqrt{2}\sqrt{\frac{[vv]}{n-1}} \\ r_0 &= \rho\sqrt{2}\sqrt{\frac{[vv]}{n(n-1)}} \end{aligned}$$

where $\rho\sqrt{2} = 0.6745$ nearly.

46. If we consider the m. s. e. μ_0 of the arithmetic mean of n observations as the true error of the mean V, the results $\mu_0 = \frac{\mu}{\sqrt{n}}$ and $\mu^2 = \frac{[vv]}{n-1}$ may be derived very neatly as follows:

We have

$$\begin{aligned} T &= M_1 + \Delta_1 \\ T &= M_2 + \Delta_2 \\ &\cdot \quad \cdot \quad \cdot \\ T &= M_n + \Delta_n \end{aligned}$$

By addition, taking the mean

$$\begin{aligned} T &= \frac{1}{n}(M_1 + M_2 + \ldots + M_n) + \frac{1}{n}(\Delta_1 + \Delta_2 + \ldots + \Delta_n) \\ &= V + \frac{1}{n}(\Delta_1 + \Delta_2 + \ldots + \Delta_n) \end{aligned}$$

$$\therefore \qquad \text{Error of } V = \frac{1}{n}(\varDelta_1 + \varDelta_2 + \ldots + \varDelta_n)$$

and

$$\mu_0^2 = \frac{1}{n^2}(\varDelta_1 + \varDelta_2 + \ldots + \varDelta_n)^2$$

$$= \frac{[\varDelta\varDelta]}{n^2}$$

But from Art. 20

$$\mu^2 = \frac{[\varDelta\varDelta]}{n}$$

$$\therefore \ \mu_0 = \frac{\mu}{\sqrt{n}}$$

Again,

$$\varDelta = T - M$$

$$= V \pm \frac{\mu}{\sqrt{n}} - M$$

$$= v \pm \frac{\mu}{\sqrt{n}}$$

and

$$\mu^2 = \frac{[\varDelta\varDelta]}{n}$$

$$= \frac{1}{n}\left[\left(v \pm \frac{\mu}{\sqrt{n}}\right)^2\right]$$

Hence, remembering that $[v] = 0$,

$$\mu^2 = \frac{[vv]}{n-1}$$

47. (b) *Peters' Formula.*

The m. s. e. and p. e. of a series of observed values may be more rapidly computed from the sum of the errors rather than from the sum of their squares by means of the convenient formula first given by Dr. Peters.*

From the equation

$$[vv] = \frac{n-1}{n}[\varDelta\varDelta]$$

* *Astronomische Nachrichten*, No. 1034.

we have approximately, without regard to sign,

$$v_1 = \sqrt{\frac{n-1}{n}}\,\Delta_1,$$

$$v_2 = \sqrt{\frac{n-1}{n}}\,\Delta_2,$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

Adding and dividing by n,

$$\frac{[v]}{n} = \sqrt{\frac{n-1}{n}}\,\eta$$

But from Art. 22

$$\mu = \sqrt{\frac{\pi}{2}}\,\eta$$

$$\therefore\ \mu = \sqrt{\frac{\pi}{2n(n-1)}}\,[v]$$

$$= \frac{1.2533}{\sqrt{n(n-1)}}\,[v] \quad \text{nearly.}$$

For a demonstration of this formula more rigorous in form see *Astron. Nachr.*, No. 2039.

As in Art. 23, it follows that the precision of this formula is expressed by the complete form

$$\mu = \frac{1.2533}{\sqrt{n(n-1)}}\,[v]\left\{1 \pm \sqrt{\frac{\pi-2}{2(n-1)}}\right\}$$

the last term giving the mean uncertainty of μ.

For the p. e. of an observation and of the arithmetic mean of n observations we have respectively

$$r = \rho\sqrt{2}\,\mu = \frac{0.8453}{\sqrt{n(n-1)}}\,[v]$$

$$r_0 = \frac{0.8453}{n\sqrt{n-1}}\,[v]$$

The mode of deriving Peters' formula given in the preceding is approximate, and the formula itself is not very

close for small values of n. Thus when $n = 2$, if d denotes the difference of the observed values, then by Bessel's formula $\mu = \frac{d}{\sqrt{2}}$, and by Peters' formula $\mu = 1.25\frac{d}{\sqrt{2}}$, which is one-fourth greater. The corresponding probable errors are in the same ratio.

A formula for the probable error of the mean which answers better than Peters' for small values of n has been derived by Fechner (Poggendorff, *Annalen*, *Jubelband*, 1874) as follows:

$$r_o = \frac{1.1955}{\sqrt{2n - 0.8548}} \frac{[v}{n}$$

As this formula is troublesome to compute, and as it gives results agreeing closely with those found from Peters' formula when n is a moderately large number, there is no advantage to be derived from the use of it in ordinary work.

48. Collecting the formulas for finding the p. e. of a single observation and of the arithmetic mean of n observations, we have

$$r = 0.6745\sqrt{\frac{[vv]}{n-1}} \qquad r = 0.8453\frac{[v}{\sqrt{n(n-1)}}$$

$$r_o = 0.6745\sqrt{\frac{[vv]}{n(n-1)}} \qquad r_o = 0.8453\frac{[v}{n\sqrt{n-1}}$$

To save labor in the numerical work I have computed tables containing the values of the coefficients of $\sqrt{[vv]}$ and $[v$ in these equations for values of n from 2 to 100. (See Appendix, Tables II., III.)

If Bessel's formula is used compute first $[vv]$, then $\sqrt{[vv]}$ can be taken from a table of squares closely enough. This square-root number multiplied by the number in Table II. corresponding to the given value of n gives the p. e. sought. If Peters' formula is used multiply the sum of the residuals, without regard to sign, by the numbers in Table III. corresponding to the argument n.

49. Control of $[vv]$.—A control is afforded by the derivation of $[vv]$ from the observed values and the arithmetic mean directly.

We have

$$\begin{aligned} v_1 &= V - M_1 \\ v_2 &= V - M_2 \\ \cdot \quad & \cdot \quad \cdot \quad \cdot \\ v_n &= V - M_n \end{aligned}$$

Square and add,

$$\begin{aligned} [vv] &= nV^2 - 2V[M] + [M^2] \\ &= [M^2] - [M]V, \qquad\qquad (1) \end{aligned}$$

$$\text{since } nV = [M].$$

The values of M^2 may be found from a table of squares or from Crelle's tables, or, if the numbers M are large, an arithmometer, or machine for multiplying and dividing, may be employed with advantage.

The computation may often be much abbreviated by the artifice of Art. 41. Substituting the values of $M_1, M_2 \ldots M_n$ from that article in (1), we find, after a simple reduction,

$$\begin{aligned} [vv] &= [ll] - \frac{[l]^2}{n} \\ &= [ll] - [l]v' \end{aligned}$$

50. Approximate Method of Finding the Precision.—A connection between the p. e. of a single observation and the greatest error committed in the series may be established approximately by the aid of the principle proved in Art. 30. There we saw that in a large series the actual errors may be expected to range between zero and 4 or 5 times the p. e. of an observation. If, then, we find from the observations a p. e. of an amount, say, r, we may assert that the greatest actual error is not likely to be more than $5r$. The probability of its being as large as this is only about $\frac{1}{1000}$.

The same principle will enable us to estimate roughly the p. e. in a series of observations. A glance at the

measured results will show the largest and smallest, and their difference may be taken as the range in the results, and half the difference as the maximum error. Hence, since in an ordinary series of from 25 to 100 observations the maximum error may be expected to be from 3 to 4 times the p. e., we may take *the p. e. to be from $\frac{1}{6}$ to $\frac{1}{8}$ of the range of the errors of observation.*

This result may be confirmed as follows: Expanding the exponential function $\varphi(\Delta)$ in a series, we may write

$$\varphi(\Delta) = P - Q\Delta^2 + R\Delta^4 - \ldots$$

where $P, Q, R, \ldots$ are constants.

If a denotes the maximum error, then, since the probability of the occurrence of an error between the limits $+a$ and $-a$ is certainty, we have, taking two terms of the series,

$$\int_{-a}^{+a} (P - Q\Delta^2)\, d\Delta = 1$$

Also, since a is the maximum error, its probability is zero.

$$\therefore P - Qa^2 = 0$$

From these two equations P and Q may be found in terms of a.

To find the p. e. r we have by definition (see Art. 21).

$$\int_{-r}^{+r} (P - Q\Delta^2)\, d\Delta = \frac{1}{2}$$

or

$$Pr - \frac{1}{3}Qr^3 = \frac{1}{2}$$

Substituting for P and Q their values, and solving for r, we find

$$r = \frac{a}{3} \text{ nearly}$$

that is, *the p. e. is approximately $\frac{1}{3}$ of the maximum error, or $\frac{1}{6}$ of the range of the errors of observation.*

A closer approximation would be found by taking three terms of the series for $\varphi(\Delta)$. We should then find

$$r = \frac{a}{4} \text{ nearly.}$$

See note by Capt. Basevi, R.E., in *G. T. Survey of India*, vol. iv.; also Helmert in *Zeitschr. für Vermess.*, vol. vi.

Ex. We shall now apply the preceding formulas to the example in Art. 42 to find the m. s. e. and p. e. of the arithmetic mean and of a single observation.

(1) *The m. s. e. and p. e. of the arithmetic mean.*

These we may find in two ways:

(a) From the sum of the squares of the residuals (Art. 43):

$$\mu_0 = \sqrt{\frac{[vv]}{n(n-1)}}$$

$$= \sqrt{\frac{0.2756}{21 \times 20}}$$

$$= 0.026$$

$$r_0 = 0.6745 \times 0.026$$

$$= 0.017$$

or from Table II. at once:

$$r_0 = 0.525 \times 0.033$$

$$= 0.017$$

(b) From the sum $[v$ of the residuals (Art. 47):

The multiplier in Table III. corresponding to the number 21 is 0.009.

$$\therefore r_0 = 2.02 \times 0.009$$

$$= 0.018$$

(2) *The p. e. of a single observation.*

From Tables II. and III. directly:

$$r = 0.525 \times 0.151 = 0.079$$

$$r = 2.02 \times 0.041 = 0.082$$

Check (α). Let the residuals be arranged in order of magnitude. They are:

0.23 0.22 0.19 0.16 0.16 0.11 0.11 0.10 0.10 0.10 0.09
0.09 0.06 0.06 0.05 0.04 0.04 0.04 0.04 0.02 0.01

The residual 0.09 occupies the middle place, and is therefore the p. e. r of a single result (Art. 21). The computation above gives 0.08.

Check (β). See Art. 50.

$$\text{Range} = 0.22 + 0.23 = 0.45$$

$$\therefore r = \frac{0.45}{6} = 0.08.$$

The values found by the different methods agree reasonably well.

51. **The Law of Error Tested by Experience.**—We shall now test our example and see how closely it conforms to the law of error, and hence be in a better position to judge of how far the law of error itself is applicable in practice. This is the *à posteriori* proof intimated in Art. 11 as necessary for the demonstration of the law.

(1) The number of + residuals is 12, and the number of − residuals is 9.

(2) The sum of the + residuals is 1.03, and the sum of the − residuals is 0.99.

(3) The sum of the squares of the + residuals is 1417, and of the − residuals is 1339.

(4) The p. e. of a single observation is 0.08. To find the number of observations we should expect whose residual errors are not greater than 0.10, we enter Table I. with the argument $\frac{0.10}{0.08} = 1.25$ and find 0.60. This multiplied by 21 gives 13 as the number of errors to be expected not greater than 0.10. By actual count we find the number observed to be 14.

To find the number to be expected between 0.10 and 0.20 we enter the table with the argument $\frac{.20}{.08} = 2.50$ and find 0.91. From this deduct 0.60 and multiply the remainder by 21. This gives 6. The number observed is 5.

The number to be expected over 0.20 is, by theory, 2. The number observed is 2.

The preceding results are collected in the following table:

Limits of Error.	Number of Errors.	
	Theory.	Observation.
s. s. 0.00 to 0.10	13	14
0.10 to 0.20	6	5
over 0.20	2	2

Table I., it will be remembered, is founded on the supposition that the number of observations in a given set is very large. In our example the number is only 21. Perfect accordance between the number of errors given by theory and the number given by observation is, therefore, not to be expected. For longer examples of this kind see Chauvenet's *Least Squares*, p. 489; Airy's *Theory of Errors of Observations*, third ed., appendix.

Comparisons between the number of errors within given limits that actually occur in a series of observations and the number to be expected from theory in the same series show the degree of confidence we may place in the law of error. It is the final criterion, and forms the second part of the proof of the law as stated in Art. 11.

The law of error has been so thoroughly tested in this way, so far as the sciences of observation (for which, indeed, it was framed) are concerned, that if in a series of observations we find that the errors do not conform to it we may suspect the presence of other than accidental sources of error. For example, Bessel * found, from his reduction of a series of 300 observations made by Bradley on the declinations of α Tauri, etc., that the numbers of errors that actually occurred and the numbers given by theory within specified limits were as follows:

Limits.	Number of Errors.	
	Experience.	Theory.
0″.0 to 0″.4	66	65
0″.4 to 0″.8	58	60
0″.8 to 1″.2	55	53
1″.2 to 1″.6	28	41
1″.6 to 2″.0	27	30
2″.0 to 2″.4	21	21
2″.4 to 2″.8	10	13
2″.8 to 3″.2	15	8
3″.2 to 3″.6	8	5
3″.6 to 4″.0	4	2
over 4″	6	2

* *Fundamenta Astronomiae*, [illegible]

The agreement, especially for the larger errors, is not very close. Prof. Safford, to whom I referred this example, states that in a new reduction of Bradley's observations made by Auwers a much better agreement between experience and theory is found. Three sources of systematic error enter into the observations which were not taken into account in Bessel's reduction:

(a) Personal equation. Bradley was not the only observer, Mason and Green being the others.

(b) Local deviation of the plumb-line.

(c) The assumption that the collimation of Bradley's Quadrant did not vary irregularly.

Auwers has in a great degree overcome these difficulties in his reduction. He finds, for example, the number of errors over 4″ to be 2 instead of 6, thus agreeing with the theoretical number.

For another illustration see Helmert's discussion of the errors in Koppe's triangulation for determining the axis of the St. Gothard Tunnel. (*Zeitschr. für Vermess.*, vol. v. pp. 146 seq.)

52. As the Gaussian law of error is found to apply reasonably well to other phenomena, such, for example, as statistical questions, guesses, etc., it has been often rashly assumed to be of universal application; and when prediction and experience are found not to agree, the validity of the law in any case has been as rashly impugned.

In the fundamental investigation in Art. 15 the hypotheses there made are satisfied by other functions of the measured values besides the arithmetic mean. Thus, for example, taking the geometric mean g, the resulting law of error would be of the form

$$\varphi\left(\frac{\Delta}{g}\right) = ce^{-h^2 \log^2 \frac{\Delta}{g}}$$

where c and h are constants.

This form appears to apply to many statistical questions * better than the Gaussian law. That it does so is, how-

* Galton, *Proc. Roy. Soc. Lond.*, 1879, pp. 365 seq.

ever, no argument against the Gaussian law in its own territory.

A curious illustration of the preceding remarks, in the misapplication of the Gaussian law of error to a case where it would appear at first sight that it ought to apply directly, is to be found in the essay on "Target-Shooting" in Sir J. F. W. Herschel's *Familiar Lectures on Scientific Subjects.*

53. **Caution as to the Application of the Tests of Precision.**—In the preceding article we have given several cautions with regard to the strict application of the law of error in practice. We shall now perform a similar service for the tests of precision—the m. s. e. and the p. e.

The m. s. e. and p. e. of series of observations have been defined as measures of their relative accuracy. With ideal series—that is, series which do not contain systematic errors, and in which the accidental errors are continuous in magnitude between the extreme limits of error—this is true; and in actual series, in good work, it is on the whole true. But in any actual series selected at random we must apply these tests with caution. It is a common mistake to overlook the distinction between observations which conform to the law of error and those which only apparently do so, and hence to condemn the m. s. e. and p. e. as not only worthless but misleading.

For example, in levelling, if the same line is run over in duplicate in the same direction, a good agreement may be expected at the several bench-marks where comparisons are made. The m. s. e. of observation will consequently be small. If the line is levelled in opposite directions experience shows that the agreement would not be so good. The m. s. e. would be larger than before. We might, therefore, hastily conclude that the first work would give the better result. But when we reflect that the main differences arise from such causes as the refraction of light, the personal bias of the observer, etc., which causes are less likely to be mutually destructive and more likely to be cumulative if

the lines are run in the same direction, it is to be expected that the final result obtained from measurements in opposite directions will be nearer the truth. The conclusion arrived at by trusting to the m. s. e. alone would be illusory.

Again, in measuring a horizontal angle, if the same part of the limb of the instrument is used in making the readings the results may be very accordant and the m. s. e. consequently small, but the angle itself may not be anywhere near the truth. This would be shown, for example, by the large discrepancies in the sums of angles of triangles measured in this way from the theoretical sums. For a long time this contradiction was a source of much perplexity, and many good instruments were most unjustly condemned.* At last the discovery was made that it was mainly owing to periodic errors of graduation of the limb which, when corrected for, made the remaining errors fairly subject to the law of error. This most important discovery may be said to have revolutionized the art of measuring horizontal angles.

The difficulty may be explained in this way. In the derivation of the m. s. e. from a series of n observed quantities M_1, M_2, . . . we had the observation equations

$$\begin{aligned} V - M_1 &= v_1 \\ V - M_2 &= v_2 \\ &\cdot\;\cdot\;\cdot\;\cdot \\ V - M_n &= v_n \end{aligned}$$

Also
$$\mu^2 = \frac{[vv]}{n - 1}$$

Now, if we suppose each of the observed quantities to be changed by the same amount c, which may be of the nature of a constant error or correction, so that they become $M_1 + c$, $M_2 + c$, . . . the most probable value, instead of being V, will be $V + c$. Also since

$$\begin{aligned} v &= (V + c) - (M + c) \\ &= V - M \end{aligned}$$

the residual errors will be the same as before.

* See, for example, *G. T. Survey of India*, vol. ii. pp. 51, 96.

Hence μ^2 is unchanged, and we see, therefore, that the m. s. e. makes no allowance for constant errors or corrections to the observed quantity. These are supposed to be eliminated or corrected for before the most probable value and its precision are sought.

Another common misapprehension is the following: From Arts. 19 or 46 the relation between the m. s. e. of a single observation μ and the m. s. e. of the mean of n observations μ_0 is

$$\mu_0 = \frac{\mu}{\sqrt{n}}$$

This formula shows that by repeating the measurement a sufficient number of times we can make the m. s. e. of the final result as small as we please. Nothing would, therefore, seem to be in the way of our getting an exact result, and that we could do as good work with a rude or imperfect instrument as with a good one by sufficiently increasing the number of observations.

Experience, however, shows that in a long series of measurements we are never certain that our result is nearer the truth than the smallest quantity the instrument will measure. If an instrument measures seconds we cannot be sure that by repeating the observations we can get the nearest hundredth or tenth. In a word, we cannot measure what we cannot see.

Take an example: With the meridian circle Prof. Rogers found the p. e. of a single complete observation in declination to be $\pm$ 0″.36, and the p. e. of a single complete observation in right ascension for an equatorial star to be $\pm$ 0ˢ.026. He says: "If, therefore, the p. e. can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining from a moderate number of observations the right ascension within 0ˢ.02 and the declination within 0″.2. Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute co-ordinates are

known within these limits."* The explanation is, as intimated, that constant errors are not eliminated by increasing the number of observations. Accidental errors are eliminated by so doing; and if a number of observations expressed by an infinity of a sufficiently high order could be taken, so that the constant errors entering in the different series could be classed as accidental, these errors would mutually balance in the reduction and we should arrive at the true result.

Closely allied to the preceding is the common idea that if we have a poor set of observations good results can be derived from them by adjusting them according to the method of least squares, or that if work has been coarsely done such an adjustment will bring out results of a higher grade. A seeming accuracy is obtained in this way, but it is a very misleading one. The method of least squares is no philosopher's stone: it has no power to evolve reliable results from inferior work.

A third source of uncertainty from the same cause may be mentioned. It may happen that the value obtained of the p. e. is numerically greater than that of the observed quantity itself. It is then a question whether in subsequent investigations we should use the value of the observed quantity as found or neglect it. This depends on circumstances. It is ever a principle in least squares to make use of all the knowledge on hand of the point at issue. If we have strong *à priori* reasons for expecting the value zero it would be better to take this value. Thus if we ran a line of levels between two points on the surface of a lake we should expect the difference of height to be zero. If the p. e. of the result found were greater than the result itself it would be allowable in this case to reject the determination. On the other hand, when we have no *à priori* knowledge, as in determinations of stellar parallax, for example, if the p. e. of the value found were in excess of the value itself, as is sometimes the case,† we could do nothing but

* *Proc. Amer. Acad. Sci.*, 1878, p. 174. † See, for example, Newcomb, *Astronomy*, app. vii.

take the value resulting from the observations, unless, indeed, it came out with a negative sign, and then its unreliable character would be evident.

54. **Constant Error.**—The remarks on constant error in the preceding articles lead us to notice an example or two of the detection and treatment of this great bugbear of observation.

We suspect the presence of constant errors in a series of observations from the large range in the results—a range greater than would naturally be expected after all known corrections have been applied. Great caution is necessary in dealing with such cases, and one should be in no hurry to jump at conclusions.

Sometimes the sources of error are detected without much trouble. Thus in measuring an angle with a theodolite, if the instrument is placed on a stone pillar firmly embedded in the ground, the range in results, if targets are the signals pointed at, would not usually be over 10″ in primary work; and on reading to a number of signals in order round the horizon the final reading on closing the horizon would be nearly the same as the initial reading on the same signal. If next the instrument were placed on a wooden post or tripod, and readings made to signals in order round the horizon in the same way as before, the final reading might differ from the initial by a large amount. The observations might also show that the longer the time taken in going around the greater the resulting discrepancy. The natural interence would be that in some way the wooden post had to do with the discrepancy in the results. In an actual case* of this kind examination showed the change to be most uniform on a day when the sun shone brightly. Measurements were then made at night, using

* At U. S. Lake Survey station Brulé, Lake Superior, many observations were taken during both day and night in July, 1871, to determine the rate of twist of centre-post on which the theodolite used in measuring angles was placed. The conclusion arrived at was that "during a day of uniform sunshine and clear atmosphere this twist seemed to be quite regular, and at the rate of about *one second of arc per minute of time*, reaching a maximum about 7 P.M. and a minimum about 7 A.M., during the month of July. On partially cloudy days there was no regularity in the twist, being sometimes in one direction and again in the opposite."

lamps as signals on the distant stations, and the same change was observed, only it was in the opposite direction.

The effect on the value of an angle of this twist of station, assuming it to act uniformly in the same direction during the time of observation, can be eliminated by the method of observation: first reading to the signals in one direction and then immediately in the opposite direction, and calling the mean of the difference of the two sets of readings a single value of the angle. So also in azimuth work the mean of the difference of the readings, star to mark, and mark to star, gives a single value free from station-twist.

This mode of procedure is in accordance with the general principle to eliminate a correction, when possible, by the method of observation, rather than to compute and apply it. See Art. 2.

The effort to avoid systematic error causes in general a considerable increase of labor, and sometimes this is very marked. For example, in the micrometric comparison of two line measures belonging to the U. S. Engineers the results found by different observers showed large discrepancies. The micrometer microscopes used were of low power, with a range of about one mm. between the upper and lower limits of distinct vision. Examination showed that the discrepancies arose mainly from focusing, each observer's results being tolerably constant for his own focus. As the value of a revolution of the micrometer screw entered into the reduction of the comparison work, and as this value was obtained from readings on a space of known value, error of focusing entered from this source. Hence a value of the screw had to be determined from a special set of readings taken at each adjustment, and this value used in reducing the regular observations made with the same focus. Had the microscopes been of high power it would have been sufficient to have determined the value of the screw once for all, since the error arising from change of focus could have been classed as accidental.

In trying to avoid systematic error the observer will, as he gains in experience, take precautions which would at first seem to be almost childish. Good work can only be had at the cost of eternal vigilance.

55. **Necessary Closeness of Computation.**—The number of observations necessary to the proper determination of a quantity may be approximated to by referring to the m. s. e. of these observations. If, after planning the method of observing with the view of eliminating "constant error," we find that increasing the number of observations beyond a certain limit does not sensibly affect the m. s. e., we may conclude that we have a sufficient number. It is evident that increasing or decreasing this number will affect the result more or less. But we cannot say that we are nearer the truth in either case.

Hence the folly of a too rigorous computation. An approximate value being the best that we can have at any rate, no weight is added to it by carrying out that value to a great many decimal places. Thus if the most probable value of a quantity computed in an approximate way is V, whereas the value found from the same observations by a rigorous computation is $V+c$, we may estimate the allowable value of c as follows. Let $\varDelta_1$, $\varDelta_2$, . . . $\varDelta_n$ be the errors of V, then $\varDelta_1+c$, $\varDelta_2+c$, . . . $\varDelta_n+c$ are the errors of $V+c$. Hence

$$\mu_{V+c}^2 = \frac{1}{n}\left\{(\varDelta_1+c)^2+(\varDelta_2+c)^2+\ldots+(\varDelta_n+c)^2\right\}$$

$$= \frac{[\varDelta\varDelta]}{n}+c^2$$

$$= \mu_V^2+c^2$$

and

$$\mu_{V+c} = \mu_V\left(1+\frac{1}{2}\frac{c^2}{\mu_V^2}\right) \text{ approximately.}$$

Now, we may safely allow the difference between μ_{V+c} and

μ_V to be $\frac{\mu_V}{100}$. Hence there will be no appreciable error introduced by computing in such a way that

$$\frac{1}{2}\frac{c^2}{\mu_V^2} < \frac{1}{100}$$

that is, when the error c committed is, roughly, $\frac{1}{10}$ of the m. s. e.

B. *Observed Values of Different Quality.*

56. **The Most Probable Value: the Weighted Mean.**—It has been shown in Art. 17 that if the directly observed values $M_1, M_2, \ldots M_n$ of a quantity are of different quality, the most probable value is found by multiplying each residual error of observation by the reciprocal of its m. s. e., and making the sum of the squares of the products a minimum; that is, with the usual notation,

$$\frac{v_1^2}{\mu_1^2} + \frac{v_2^2}{\mu_2^2} + \ldots + \frac{v_n^2}{\mu_n^2} = \text{a min.} \qquad (1)$$

or

$$\left(\frac{V - M_1}{\mu_1}\right)^2 + \left(\frac{V - M_2}{\mu_2}\right)^2 + \ldots + \left(\frac{V - M_n}{\mu_n}\right)^2 = \text{a min.} \qquad (2)$$

By differentiation and reduction,

$$V = \left[\frac{M}{\mu^2}\right] \div \left[\frac{1}{\mu^2}\right] \qquad (3)$$

We have, therefore, the equivalent rule:

If the observed values of a quantity are of different quality, the most probable value is found by multiplying each observed value by the reciprocal of the square of its m. s. e., and dividing the sum of the products by the sum of the reciprocals.

The form of the expression for V suggests another stand-point from which to consider it. Let $p_1, p_2, \ldots p_n$

be the numerical parts of $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \ldots \frac{1}{\mu_n^2}$, such that each is of the type

$$p = \frac{(\text{unit of measure})^2}{\mu^2}$$

then equation (3) may be written

$$V = \frac{[pM]}{[p]} \tag{4}$$

Also, since $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \ldots \frac{1}{\mu_n^2}$ are similarly involved in the numerator and denominator of the value of V, this value will remain the same if $p_1, p_2, \ldots p_n$ are taken any numbers whatever in the same proportion to $\frac{1}{\mu_1^2}, \frac{1}{\mu_2^2}, \ldots \frac{1}{\mu_n^2}$; that is, if $p_1, p_2, \ldots p_n$ satisfy the relations

$$p_1 = \frac{\mu^2}{\mu_1^2}, p_2 = \frac{\mu^2}{\mu_2^2}, \ldots, p_n^2 = \frac{\mu^2}{\mu_n^2} \tag{5}$$

where μ is an arbitrary constant.

57. Let us look at this question from another point of view. It is in accordance with our fundamental assumptions that observations of a quantity made under the same conditions, so that there is no *à priori* reason for choosing one before another, are of the same quality. They require the same expenditure of time, labor, money, etc. If, therefore, we represent the quality of a single observation of a certain series by unity, the quality of the arithmetic mean of p such observations, as it would require p times the expenditure to attain it, would be represented by p. Let us suppose, then, that the arithmetic mean of p_1 observed values of a certain quality is M_1; of p_2 other observed values of the same quality it is M_2, and so on. The total number of observed values is $[p]$. All of the observed values being of the same quality, the most probable value

V of the unknown is given by the arithmetic mean; that is, by

$$V = \frac{\text{sum of the values of all the sets}}{\text{sum of number of obs. in each set}}$$

$$= \frac{p_1M_1 + p_2M_2 + \ldots + p_nM_n}{p_1 + p_2 + \ldots + p_n}$$

$$= \frac{[pM]}{[p]}$$

which is of the same form as Eq. 4, above.

The numbers $p_1, p_2, \ldots p_n$ are called the *weights*, or, better, the *combining weights*, of the observed values, and the mean value V is called the *weighted mean*.

In view of this definition, the general principle stated in Art. 56 may be replaced by the following necessarily equivalent one:

If the observed values of a quantity are of different weights, the most probable value is found by multiplying the square of each residual error of observation by its weight, and making the sum of the products a minimum.

Thus the most probable value V is found from

$$[pvv] = \text{a min.}$$

that is, from

$$p_1(V - M_1)^2 + p_2(V - M_2)^2 + \ldots + p_n(V - M_n)^2 = \text{a min.}$$

By differentiation,

$$p_1(V - M_1) + p_2(V - M_2) + \ldots + p_n(V - M_n) = 0$$

whence

$$V = \frac{[pM]}{[p]}$$

This form of the value of V leads to the rule:

If the observed values of a quantity are of different weights, the most probable value is found by multiplying each observed value by its weight, and dividing the sum of the products by the sum of the weights.

As in the case of the arithmetic mean (Art. 40), it is evidently simpler in practice to find the weighted mean directly by this rule rather than from the minimum equation.

If the observed values M are numerically large we may lighten the arithmetical work by finding V by the method of Art. 41. Proceeding as there indicated, we have

$$V = X' + \frac{[pl]}{[p]}$$
$$= X' + x'' \text{ suppose}$$

58. **Reduction of Observed Values to a Common Standard.**—The principle of the weighted mean is evidently an extension of that of the arithmetic mean, as was pointed out long ago by Cotes, Simpson, and others. It merely amounts to finding a mean of several series of means, the unit of measure being the same in each. As soon, therefore, as results of different weights are changed into others having a common standard of weight, the rules for combining and finding the precision of observed quantities of the same weight can be applied to weighted quantities.

This change we are enabled to make by means of the relation (5), Art. 56, which may be written

$$\mu_1 = \frac{\mu}{\sqrt{p_1}},\ \mu_2 = \frac{\mu}{\sqrt{p_2}},\ \ldots\ \mu_n = \frac{\mu}{\sqrt{p_n}}$$

Now, since $\mu_1, \mu_2, \ldots \mu_n$ are the m. s. e. of $M_1, M_2, \ldots M_n$, the m. s. e. of $M_1\sqrt{p_1}, M_2\sqrt{p_2}, \ldots M_n\sqrt{p_n}$ would each be the same quantity μ.

Hence if a series of observed values $M_1, M_2 \ldots M_n$ have the weights $p_1, p_2, \ldots p_n$, they are reduced to the same standard by multiplying by $\sqrt{p_1}, \sqrt{p_2}, \ldots \sqrt{p_n}$ respectively.

For example, given the observation equations,

$$\begin{array}{lll} V - M_1 = v_1 & \text{weight} & p_1 \\ V - M_2 = v_2 & \text{"} & p_2 \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot & & \\ V - M_n = v_n & \text{"} & p_n \end{array}$$

to find the most probable value of V.

Reducing to the same standard of weight, we have the equations

$$\sqrt{p_1}\, V - \sqrt{p_1}\, M_1 = \sqrt{p_1}\, v_1$$
$$\sqrt{p_2}\, V - \sqrt{p_2}\, M_2 = \sqrt{p_2}\, v_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\sqrt{p_n}\, V - \sqrt{p_n}\, M_n = \sqrt{p_n}\, v_n$$

and the most probable value of V is found by making

$$(\sqrt{p_1}\, v_1)^2 + (\sqrt{p_2}\, v_2)^2 + \dots + (\sqrt{p_n}\, v_n)^2 = \text{a min.}$$

that is, by making

$$(\sqrt{p_1}\, V - \sqrt{p_1}\, M_1)^2 + (\sqrt{p_2}\, V - \sqrt{p_2}\, M_2)^2 + \dots + (\sqrt{p_n}\, V - \sqrt{p_n}\, M_n)^2 = \text{a min.}$$

Reducing this equation, we find, as before,

$$V = \frac{[pM]}{[p]}$$

59. **Computation of the Weights.**—If $p_1, p_2, \dots p_n$ represent a series of weights corresponding to the m. s. e. $\mu_1, \mu_2, \dots \mu_n$, then we have the relations

$$p_1\mu_1^2 = p_2\mu_2^2 = \dots = p_n\mu_n^2 = \mu^2$$

where the value of μ is entirely arbitrary. The combining weights are, therefore, known when the m.s.e. $\mu, \mu_1, \mu_2, \dots \mu_n$ are known.

These relations suggest that it would be convenient to define μ as the m. s. e. of a single observation assumed to be of the weight unity.

We shall define μ in this way, so that in future it is understood that the *standard* to which observations of different weights are reduced for comparison and combination is the fictitious observation whose weight is unity and whose m. s. e. is μ.

60. **Control of the Weighted Mean.**—Eq. 2, Art. 57, may be written

$$[pv] = 0$$

Hence if the weighted mean V has been computed correctly, the sum of the products of each residual error by its weight is equal to zero.

Usually, however, V is not an exact quotient—that is, $[pM]$ is not exactly divisible by $[p]$—and then the discrepancy of $[pv]$ from zero is evidently equal to the product of the sum of the weights and the difference between the exact value of V and the approximate value used. See Art. 42.

Ex. 1.—Deduce the relation $[pv] = 0$ from the observation equations directly.

[Multiply each observation equation by its weight, and add the products.]

Ex. 2.—Find the most probable value of the velocity of light from the following determinations by Fizeau and others:

298000	kil.	± 1000	kil.
298500	"	± 1000	"
299990	"	± 200	"
300100	"	± 1000	"
299930	"	± 100	"

(*Amer. Jour. Sci.*, vol. xix.)

The weights, being inversely as the squares of the probable errors, are as the numbers 1, 1, 25, 1, 100. (Art. 59.)

(a) *Direct solution.*

M	p	pM
298000	1	298000
298500	1	298500
299990	25	7499750
300100	1	300100
299930	100	29993000
	128	38389350

$$\therefore\ V = \frac{[pM]}{[p]} = 299917 \text{ kil. approx.},$$

the exact value being $299916\frac{61}{64}$ kil.

(b) *Solution according to Art.* 57.

Assume $X' = 298000$

l	p	pl
0	1	0
500	1	500
1990	25	49750
2100	1	2100
1930	100	193000
	128	245350

$$\therefore x'' = \frac{[pl]}{[p]} = 1916\tfrac{51}{64}$$

and $V = 298000 + 1916\frac{51}{64} = 299916\frac{51}{64}$, as before.

Control. Take $V = 299917$, and proceed to find $[pv]$. (See Art. 60.)

v	p	pv
1917	1	1917
1417	1	1417
$-$ 73	25	$-$ 1825
$-$ 183	1	$-$ 183
$-$ 13	100	$-$ 1300
		26

The discrepancy should be

$$128\,(299917 - 299916\tfrac{51}{64}) = 26$$

which it is.

61. **The Precision of the Weighted Mean.**—Since the weighted mean V is the arithmetic mean of $[p]$ observations of the unit of weight, its weight is $[p]$. Hence the m. s. e. μ_V of V is found from

$$\mu_V^2 = \frac{\mu^2}{[p]}$$

where μ is the m. s. e. of an observation of the unit of weight (standard observation).

According to Art. 58, the value of μ may be found by writing $\sqrt{p_1}\,v_1$, $\sqrt{p_2}\,v_2$, . . . for v_1, v_2, . . . in the formulas derived for observations of the same weight. Hence, substituting in Bessel's and in Peters' formulas Arts. 43, 47, we have

$$\mu^2 = \frac{[pvv]}{n-1} \qquad \mu = 1.2533\,\frac{\left[\sqrt{p}\,v\right]}{\sqrt{n(n-1)}}$$

and therefore

$$\mu_V^2 = \frac{[pvv]}{[p]\,(n-1)} \qquad \mu_V = 1.2533\,\frac{\left[\sqrt{p}\,v\right]}{\sqrt{[p]\,n(n-1)}}$$

These expressions reduce to those for the arithmetic mean where the observed values are of the same weight by putting $[p] = np$.

62. *Control of* $[pvv]$.—A control of the accuracy of $[pvv]$ is afforded by the derivation of this quantity from the observed values directly.

The observation equations are

$$\begin{array}{lll} v_1 = V - M_1 & \text{weight} & p_1 \\ v_2 = V - M_2 & \text{"} & p_2 \\ . \quad . \quad . & . & . \\ v_n = V - M_n & \text{"} & p_n \end{array}$$

Hence

$$\begin{array}{l} p_1 v_1^2 = p_1 V^2 - 2p_1 V M_1 + p_1 M_1^2 \\ p_2 v_2^2 = p_2 V^2 - 2p_2 V M_2 + p_2 M_2^2 \\ . \quad . \quad . \quad . \quad . \quad . \end{array}$$

By addition,

$$[pvv]=[p]V^2-2[pM]V+[pMM]$$

$$=[pMM]-\frac{[pM]^2}{[p]}$$

since

$$V=\frac{[pM]}{[p]}$$

In using this formula the troublesome term is $[pMM]$. With large values of M it is better to deduce it from the column pM, already computed in finding the weighted mean. This is specially advisable if one has a machine for performing multiplications. With small values of M a table of squares is best.

With large values of M we may perhaps proceed still more conveniently by the method explained in Arts. 41, 57. With the same notation and reasoning as there employed it is found that

$$[pvv]=[pll]-\frac{[pl]^2}{[p]}$$

$$=[pll]-[pl]x''$$

where the quantities l are numerically small.

Ex. The linear values found for the space $0^{in}.00$ to $0^{in}.05$ of inch $[ab]$ on the standard steel foot I F. of the G. T. Survey of India were as follows: $0^{in}.050027$, $0^{in}.049971$, $0^{in}.050019$, $0^{in}.050079$, $0^{in}.050021$, $0^{in}.050011$. The numbers of measures in these determinations were 6, 6, 15, 15, 8, 8 respectively.

Taking the numbers of measures as the weights of the respective determinations, required the most probable value of the space and its p. e.

The direct solution presents no difficulty. The value of V may be found as in Ex. Art. 60, and thence the residuals v. The m. s. e. or p. e. follows from the formulas of Art. 61.

We shall give the solution according to the methods of Arts. 57 and 62, which are advantageous in this case on account of the large numbers that enter.

Assume $X' = 0.049971$

l	p	pl	pll
56	6	336	18816
0	6	0	0
48	15	720	34560
108	15	1620	174960
50	8	400	20000
40	8	320	12800
	58	3396	$261136 = [pll]$ $198842 = [pl]x''$ $62294 = [pvv]$

$$x'' = \frac{3396}{58} = 59 \qquad r = 0.6745\sqrt{\frac{62294}{58(6-1)}}$$

and

$$V = 0.049971 + 0.000059$$
$$= 0.050030 \qquad = 0.000010$$

Hence

$$V = 0^{in}.050030 \pm 0^{in}.000010$$

By choosing the approximate value X' equal to the smallest of the measures or equal to the greatest of them, all of the remainders l have the same sign, which is a great convenience in computation.

In this example an important practical point occurs, and one often overlooked. The p. e. is not computed from the original observations, but from these observations grouped in six sets of means. These means we have treated as if original observations of certain weights. Had the original observations been accessible we should have used them, and would most probably have found a different value of the p. e. from that which we have obtained. This arises from the small number of observations in the several sets. In good work the difference to be expected between the value of the p. e. found from the means and that found from the original observations would be small. Still, whenever there is a choice, the p. e. should always be deduced

from the original observations rather than from any combinations of them.

The weighted mean value V would evidently be the same whether computed from the partial means or from the original observations.

Observed Values Multiples of the Unknown.

63. Let the observed values $M_1, M_2, \ldots M_n$ be multiples of the same unknown X; that is, be of the form $a_1X, a_2X, \ldots a_nX$, where $a_1, a_2, \ldots a_n$ are constants given by theory for each observation. The values $\frac{M_1}{a_1}, \frac{M_2}{a_2}, \ldots \frac{M_n}{a_n}$ of X may be regarded as directly observed values of unequal weight. If μ is the m. s. e. of an observation, that is, of $M_1, M_2, \ldots$, then, since the m. s. e. of $\frac{M_1}{a_1}$ is $\frac{\mu}{a_1}$, of $\frac{M_2}{a_2}$ is $\frac{\mu}{a_2}$, ... the weights of these assumed observations are proportional to $a_1^2, a_2^2, \ldots$ Hence taking the weighted mean

$$X = \frac{\frac{M_1}{a_1}a_1^2 + \frac{M_2}{a_2}a_2^2 + \ldots + \frac{M_n}{a_n}a_n^2}{a_1^2 + a_2^2 + \ldots + a_n^2}$$
$$= \frac{[aM]}{[a^2]}$$

Also, since $[a^2]$ is the weight of X,

$$\mu_X^2 = \frac{\mu^2}{[a^2]}$$

Cotes,* in solving this problem, reasons that, since for the same error of M the greater a is, the less is the error of X, we may take the coefficients a to express the relative weights of the values of X. Now, placing the coefficients $a_1, a_2 \ldots$ as weights along a straight line at distances $\frac{M_1}{a_1}, \frac{M_2}{a_2}, \ldots$ from one end of the line, the most probable

* *Harmonia Mensurarum.* Cambridge, 1722. (Quoted by Hultman, *Minsta Qvadr.* Stockholm, 1860.)

value will be the distance X of the centre of gravity of these weights from the end point, and will be found by taking moments about this point; that is,

$$X(a_1+a_2+\ldots+a_n)=a_1\frac{M_1}{a_1}+a_2\frac{M_2}{a_2}+\ \ldots\ +a_n\frac{M_n}{a_n}$$

and therefore

$$X=\frac{[M]}{[a]}$$

Ex. To test the power of the telescope of the great theodolite (3 ft.) of the English Ordnance Survey, and find the p. e. of an observation, a wooden framework was set up 12,462 ft. distant from the theodolite when at station Ben More, Scotland. It was so arranged that when projected against the sky a fine vertical line of light, the breadth of which was regulated by the sliding of a board, was shown to the observer. The breadth of this opening was varied by half-inches from $1\frac{1}{2}$ in. to 6 in. during the observations, which were as follows:*

No. of obs.	Width.	Side of opening.	Mean of micr. readings.
1	6.0	left. right.	28.00 37.50
2	5.5	l. r.	28.50 37.00
3	5.0	l. r.	29.16 37.16
4	4.5	l. r.	30.16 36.66
5	4.0	l. r.	30.50 37.16
6	3.5	l. r.	31.16 37.00
7	3.0	l. r.	32.66 36.83
8	2.5	l. r.	33.50 36.83
9	2.0	l. r.	33.83 37.00
10	1.5	l. r.	35.50 37.16

* *Account of the Principal Triangulation*, pp. 54, 55.

Let X = the most probable value of the angle subtending an opening of 1 inch. Then we have the observation equations

$$\begin{aligned} 6\,X - 9.50 &= v_1 & 3.5\,X - 5.84 &= v_6 \\ 5.5\,X - 8.50 &= v_2 & 3\,X - 4.17 &= v_7 \\ 5\,X - 8.00 &= v_3 & 2.5\,X - 3.33 &= v_8 \\ 4.5\,X - 6.50 &= v_4 & 2\,X - 3.17 &= v_9 \\ 4\,X - 6.66 &= v_5 & 1.5\,X - 1.66 &= v_{10} \end{aligned}$$

From the preceding we have for the individual values of X and their weights

$$X = 1.58 \quad \text{weight } 6^2$$
$$X = 1.55 \quad \text{weight } 5.5^2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \text{ weighted mean} = \frac{1.58 \times 6^2 + 1.55 \times 5.5^2 + \ldots}{6^2 + 5.5^2 + \ldots}$$
$$= 1.55$$

or making the sum of the squares of the residuals v a minimum, that is,

$$(6X - 9.50)^2 + (5.5\,X - 8.50)^2 + \ldots = \text{a min.}$$

we find by differentiation that

$$X = 1.55$$

as before.

The practical rule following from either method is the same, and may be stated thus: Multiply each observation equation by the coefficient of X in that equation, and add the products. The resulting equation gives the value of X.

We again see that the principle of minimum squares is more general than that of the arithmetic mean, and why it was that we failed in solving the equations of Art. 14. In other words, we cannot write in the preceding example

$$X = 1.58$$
$$X = 1.55$$
$$\cdot \quad \cdot \quad \cdot$$

and take the mean of these values as the value of x, because these equations should be written strictly

$$X = 1.58 + \frac{v_1}{6}$$

$$X = 1.55 + \frac{v_2}{5.5}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where the errors are not lessened in the same proportion throughout the equations.

Precision of a Linear Function of Independently Observed Quantities.

64. Suppose that there are n independently observed quantities M_1, M_2, . . . whose m. s. e. are μ_1, μ_2, . . . respectively, to find the m. s. e. μ of F where

$$F = a_1M_1 + a_2M_2 + \ldots + a_nM_n \qquad (1)$$

a_1, a_2, . . . a_n being constants.

This has been already solved in Art. 19, where it was shown that

$$\mu^2 = [a^2\mu^2]$$

On account of the great importance of this result we add another method of deriving it.

If Δ_1, Δ_2, . . . denote the errors of M_1, M_2, . . . we shall have the true value T of F by writing $M_1 + \Delta_1$, $M_2 + \Delta_2$, . . . for M_1, M_2, . . . in the above expression for F; that is,

$$T = a_1(M_1 + \Delta_1) + a_2(M_2 + \Delta_2) + \ldots + a_n(M_n + \Delta_n)$$

Call Δ the error of F; then, since $T = F + \Delta$, we have

$$\Delta = a_1\Delta_1 + a_2\Delta_2 + \ldots + a_n\Delta_n$$

and $\therefore$

$$\Delta^2 = a_1^2\Delta_1^2 + a_2^2\Delta_2^2 + \ldots + 2a_1a_2\Delta_1\Delta_2 + \ldots$$

Let the number of sets of M_1, M_2, . . . required to find T be n, and suppose Δ^2 summed for all the sets of values of Δ_1, Δ_2, . . . and the mean taken, then attending to Art. 20,

$$\mu^2 = a_1^2\mu_1^2 + a_2^2\mu_2^2 + \ldots + 2a_1a_2\frac{[\Delta_1\Delta_2]}{n} + \ldots \qquad (2)$$

In forming all possible values of $\Delta_1\Delta_2$, $\Delta_2\Delta_3$, . . ., the number of values being very large, there will probably be as many + as − products of each form, and we therefore assume

$$[\Delta_1\Delta_2] = [\Delta_2\Delta_3] = \ldots = 0$$

Hence

$$\mu^2 = [a^2\mu^2] \qquad (3)$$

Ex. 1. The Keweenaw Base was measured with two measuring tubes placed end to end in succession. Tube 1 was placed in position 967 times, and tube 2 966 times. Given the p. e. of the length of tube $1 = \pm 0^{in}.00034$, and of tube $2 = \pm 0^{in}.00037$, find the p. e. in the length of the line arising from the uncertainties in the length of the tubes.

$$
\begin{aligned}
[\ \text{p. e. from tube } 1 &= 967 \times 0.00034 = 0^{in}.329 \\
\text{p. e. from tube } 2 &= 966 \times 0.00037 = 0^{in}.357 \\
\therefore \text{p. e. of line } &= \sqrt{0.329^2 + 0.357^2} \\
&= 0^{in}.485 \quad]
\end{aligned}
$$

Ex. 2. In the Keweenaw Base the p. e. of one measurement of 94 tubes, deduced from the discrepancies of six measurements of these 94 tubes, was found to be $0^{in}.03$ Show that the p. e. in the length of the line of 1933 tubes arising from the same causes may be estimated at $\pm 0^{in}.136$.

$$
\begin{aligned}
[\ \text{p. e. of 1 measurement of 1 tube} &= \frac{0.03}{\sqrt{94}} \\
\text{p. e. of base of 1933 tubes} &= \frac{0.03}{\sqrt{94}}\sqrt{1933} \\
&= \pm 0.136 \quad]
\end{aligned}
$$

Attention is called to these two problems, from the importance of the principles illustrated. In Ex. 1 the p. e. of a tube was multiplied by the whole number of tubes to find the p. e. of the base from that cause, for the reason that with whatever error the tube is affected it is cumulative throughout the measurement.

In Ex. 2 the p. e. of one tube is multiplied by the square root of the number of tubes, because each measurement is independent of every other, and the errors are as likely to be in excess as in defect, and, therefore, may be expected to destroy one another in the final result.

Ex. 3. The m. s. e. of aM_1, μ being the m. s. e. of M_1 and a a constant, is equal to $a\mu$, but the m. s. e. of the sum of the a independently observed quantities $M_1, M_2 \ldots M_a$; that is, of $M_1 + M_2 + \ldots + M_a$, the m. s. e. of each being μ, is $\sqrt{a}\,\mu$. Explain.

65. If the function F whose m. s. e. is required is not in the linear form, we first reduce it to that form, as in Art. 7, and apply Eq. 3, Art. 64. Thus, if

$$F = f(M_1, M_2 \ldots M_n)$$

the true value T of F will result if we write M_1+dM_1, $M_2+dM_2, \ldots$ for $M_1, M_2, \ldots$ the differentials representing the errors of these quantities. Then

$$T=f(M+dM, M_2+dM_2, \ldots)$$

Expanding by Taylor's theorem and retaining only the first powers of the small quantities dM_1, $dM_2, \ldots$ we have

$$T=F+\frac{\partial f}{\partial M_1}dM_1+\frac{\partial f}{\partial M_2}dM_2+\ldots+\frac{\partial f}{\partial M_n}dM_n$$

or

$$\text{Error of } F=a_1dM_1+a_2dM_2+\ldots+a_ndM_n \qquad (1)$$

where

$$a_1=\frac{\partial f}{\partial M_1},\ a_2=\frac{\partial f}{\partial M_2},\ldots,\ a_n=\frac{\partial f}{\partial M_n}$$

This expression is of the same form as (1), Art. 64. Hence

$$\mu_F^2=a_1^2\mu_1^2+a_2^2\mu_2^2+\ldots+a_n^2\mu_n^2=[a^2\mu^2]$$

The still more general case of the m. s. e. of a function of quantities which are themselves functions of the same observed quantities may be readily reduced to the form of Eq. 1. The whole point is to express the error of F as a linear function of the errors of the independently observed quantities M_1, $M_2 \ldots M_n$.

It is useful to note that Eq. 1 results from differentiating the function equation directly, as has been already pointed out in Art. 7.

Ex. 1. If μ_1, μ_2 are the m. s. e. of the measured sides AB, BC of a rectangle $ABCD$, find the m. s. e. of the area of the rectangle.
[Here

$$F=M_1M_2$$

∴ by differentiation

$$dF=M_1dM_2+M_2dM_1$$

and

$$\mu_F^2=M_1^2\mu_2^2+M_2^2\mu_1^2]$$

Ex. 2. The expansions of the steel and zinc bars of tube 1 of the Repsold base apparatus of the U. S. Lake Survey for 1° Fahr. are approximately

$$S = \overset{mm.}{0.0248} \pm \overset{mm.}{0.0001}$$
$$Z = 0.0617 \pm 0.0003$$

Show that

$$\frac{S}{Z} = \frac{2}{5} \pm \frac{1}{400} \text{ nearly.}$$

[For

$$F = \frac{S}{Z}$$

$$\therefore dF = \frac{1}{Z} dS - \frac{S}{Z^2} dZ$$

and

$$(\text{p. e.})^2 = \frac{1}{Z^2}(0.0001)^2 + \frac{S^2}{Z^4}(0.0003)^2]$$

Ex. 3. The base b and the adjacent angles A, C of a triangle ABC are measured. If their m. s. e. are respectively μ_b, μ_A, μ_C, find the m. s. e. of the angle B and of the side a.

To find μ_B.

We have

$$B = 180 + \varepsilon - A - C$$

where ε denotes the spherical excess of the triangle.

Hence, A and C being independent of one another,

$$\mu_B{}^2 = \mu_A{}^2 + \mu_C{}^2$$

To find μ_a.

$$a = b \frac{\sin A}{\sin B}$$

By differentiation,

$$da = \frac{\sin A}{\sin B} db + b \frac{\sin (C - \varepsilon)}{\sin^2 B} \sin 1'' dA + a \cot B \sin 1'' dC$$

and therefore

$$\mu_a{}^2 = \frac{\sin^2 A}{\sin^2 B} \mu_b{}^2 + \frac{b^2 \sin^2 (C - \varepsilon) \sin^2 1''}{\sin^4 B} \mu_A{}^2 + a^2 \cot^2 B \sin^2 1'' \mu_C{}^2$$

Ex. 4. Given the base b and the angles A, B of a triangle with m. s. e. μ_b, μ_A, μ_B respectively, to find the m. s. e. μ_a of the side a.

We have

$$a = b \frac{\sin A}{\sin B} \tag{1}$$

This might be expanded as in the preceding example, but more conveniently as follows:

Take logarithms of both members. Then

$$\log a = \log b + \log \sin A - \log \sin B \tag{2}$$

(a) By differentiation,

$$da = \frac{a}{b}\,db + a \cot A \sin 1''dA - a \cot B \sin 1''dB$$

Hence

$$\mu_a{}^2 = \frac{a^2}{b^2}\mu_b{}^2 + a^2\cot^2 A \sin^2 1''\mu_A{}^2 + a^2\cot^2 B \sin^2 1''\mu_B{}^2 \tag{3}$$

If, as is usually assumed in practice,

$$\mu_A = \mu_B = \mu, \text{ and } \mu_b = 0$$

then

$$\mu_a = a \sin 1''\,\mu \sqrt{\cot^2 A + \cot^2 B} \tag{4}$$

(b) Using log differences as explained in Chapter I., we have by differentiating (2)

$$\delta_a\,da = \delta_b\,db + \delta_A\,dA - \delta_B\,dB \tag{5}$$

where δ_a, δ_b are the differences corresponding to one unit for the numbers a and b in a table of logarithms, and δ_A, δ_B are the differences for 1″ for the angles A and B in a table of log sines. Hence

$$\mu_a{}^2 = \left(\frac{\delta_b}{\delta_a}\right)^2\mu_b{}^2 + \left(\frac{\delta_A}{\delta_a}\right)^2\mu_A{}^2 + \left(\frac{\delta_B}{\delta_a}\right)^2\mu_B{}^2. \tag{6}$$

The two equations (3) and (6) may be used to check one another.

Ex. 5. The following example is given for the sake of showing the form of solution by the method of logarithmic differences.

In the triangulation of Lake Superior there were measured in the triangle Middle, Crebassa, Traverse Id. (ABC)

$$\angle\ A = 57°\ 04'\ 51''.4 \qquad \mu_A = 0''.30$$
$$\angle\ B = 67°\ 15'\ 39''.2 \qquad \mu_B = 0''.29$$

The side Middle-Traverse Id. as computed from the Keweenaw Base is 16894.9 yards. Taking $\mu_b = 0.05$ yd., find μ_a and μ_C

We have

$$a = b\,\frac{\sin A}{\sin B}$$

$$\therefore \log(a + da) = \log(b + db) + \log\sin(A + dA) - \log\sin(B + dB)$$

Then (see Ex. 1, Art. 7), the differences being expressed in units of the seventh decimal place,

$$\begin{aligned}
\log(b + db) &= 4.2277556 + 257\,db \\
\log\sin(A + dA) &= 9.9239892 + \ \ 14\,dA \\
\operatorname{colog}\sin(B + dB) &= \underline{0.0351398} - \quad 9\,dB \\
\therefore \log a + \delta_a\,da &= 4.1868846 + 257\,db + 14\,dA - 9\,dB \\
\text{and } 283\,da &= \quad 257\,db + 14\,dA - 9\,dB.
\end{aligned}$$

Since $\log a$ = 4.1868846, and 283 is the difference δ_a as given in the table.

Hence $\quad \mu_a^2 = \left(\frac{257}{283}\right)^2 (.05)^2 + \left(\frac{14}{283}\right)^2 (.30)^2 + \left(\frac{9}{283}\right)^2 (.29)^2$

$$= 0.0024$$

and $\quad \mu_a = \overset{yd.}{0.05}$

Also

$$\mu_C = \sqrt{\mu_A^2 + \mu_B^2}$$
$$= \sqrt{(0.29)^2 + (0.30)^2}$$
$$= 0''.42$$

Miscellaneous Examples.

66. **Examples of Mean-Square and Probable Error.**

Ex. 1. If in a theodolite read by 2 verniers the p. e. of a reading (mean of vernier readings) is 2″, show that if it is read by 3 verniers the p. e. of a reading will be a little over 1″.5, and if read by 4 verniers a little less than 1″.5.

Ex. 2. The p. e. of an angle of a triangle is r; show that the p. e. of the triangle-error is $r\sqrt{3}$, all of the angles being equally well measured.

[Error $= 180° - (A + B + C)$.]

Ex. 3. The expansion of a bar for 1° C. is $9a \pm 9r$; show that for 1° Fahr. it is $5a \pm 5r$.

Ex. 4. The length of a measuring bar at the beginning of a measurement was $a \pm r_1$. After x measures had been made it was $b \pm r_2$. Show that the length of the n^{th} measure, the length being supposed to change uniformly with the distance measured, is

$$a + \frac{n}{x}(b - a) \pm \sqrt{\left(1 - \frac{n}{x}\right)^2 r_1^2 + \frac{n^2}{x^2} r_2^2}$$

[For if da is the error of a, and db of b, then the error of $a + \frac{n}{x}(b - a)$ is $\left(1 - \frac{n}{x}\right)da + \frac{n}{x}db$, and the above p. e. follows.

It is a common mistake to write the error in the form $da + \frac{n}{x}(db - da)$, and hence to infer that the p. e. is $\sqrt{r_1^2 + \frac{n^2}{x^2}(r_1^2 + r_2^2)}$.]

Ex. 5. Prove that the p. e. of the mean of two observations whose difference is d is $0.337\ d$, and the p. e. of each observation is $0.477\ d$.

Ex. 6. The line Monadnock-Gunstock (94469 *m.*) was computed from the Massachusetts Base (17326 *m.*) through the intervening triangulation. The p. e. of the line arising from the triangulation is $\pm\ 0^{m}.317$, and the p. e. of the base is $0^{m}.0358$; find the total p. e. of the line.

$$[\text{p.e.} = \sqrt{\left(\frac{94469}{17326} \times 0.0358\right)^2 + (0.317)^2} = \pm\ 0^{m}.372]$$

Ex. 7. The Minnesota Point Base reduced to sea-level is

$$1325 \times 15 \text{ ft. bar at } 32^\circ + 11^{in}.314 \pm 0^{in}.421$$

and

$$15 \text{ ft. bar at } 32^\circ = 179^{in}.95438 \pm 0^{in}.00012$$

show that the p. e. of the base is $\pm\ 0^{in}.450$.

[p. e. $= \sqrt{(1325 \times 0.00012)^2 + (0.421)^2} = \pm\ 0^{in}.450$. We multiply $\pm\ 0^{in}.00012$ by 1325: uncertain which sign it is; but whichever it is, it is constant all the way through.]

Ex. 8. If the zenith distance ζ of a star is observed n_1 times at upper culmination, and the zenith distance ζ' of the same star is observed n_2 times at lower culmination, show that the m. s. e. of the latitude of the place of observation is

$$\frac{\mu}{2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

μ being the m. s. e. of a single observation.

$$[\text{Latitude} = 90^\circ - \tfrac{1}{2}(\zeta + \zeta')]$$

Ex. 9. Given the telegraphic longitude results,

		h.	m.	s.	s.
Cambridge west of Greenwich	=	4	44	30.99	± 0.23
Omaha west of Cambridge	=	1	39	15.04	± 0.06
Springfield east of Omaha	=		25	08.69	± 0.11

show that

Springfield west of Greenwich	=	5	58	37.34	± 0.26

$$[\text{p. e.} = \sqrt{.23^2 + .06^2 + .11^2} = 0.26]$$

Ex. 10. Given mass of earth + mass of moon $= \dfrac{1}{305879 \pm 2271}$,

and

$$\text{mass of moon} = \frac{1}{81.44}\text{ mass of earth}$$

prove

$$\text{mass of earth} = \frac{1}{309635 \pm 2299}$$

$$[\text{For } (305879 \pm 2271) \times \frac{82.44}{81.44} = 309635 \pm 2299]$$

Ex. 11. In measuring an angle suppose

r_1 = p. e. of a pointing at a signal,
r_2 = p. e. of a reading of the limb of the instrument,
e = error of graduation of the arc read on,

then, assuming that these result from the only sources of error not eliminated, show if the limb has been changed m times and n readings taken in each position, that

$$\text{p. e. of angle} = \pm\sqrt{\frac{2(r_1^2 + r_2^2)}{mn} + \frac{e^2}{m}}$$

[For one position of the limb

$$\text{p. e. of angle} = \pm\sqrt{\frac{2(r_1^2 + r_2^2)}{n} + e^2}$$

as the error of graduation remains constant throughout each set of n readings].

Ex. 12. The distance $o - 1^{mm}$ on a graduated line-measure is read with a micrometer; show that the p. e. of the mean of two results is equal to the p. e. of a single reading.

[For distance $o - 1^{mm}$

$= \frac{1}{2}\{$(first + second rdg.) at o − (first + second rdg.) at $1^{mm}\}$

$\therefore$ (p. e.)$^2 = \frac{1}{4}\{4$ (p. e.)^2 of a rdg.$\}$]

Ex. 13. In the comparison of a *mm.* space on two standards placed side by side and read with a micrometer, the p. e. of a single micrometer reading being a, show that the p. e. of the difference of the results of n combined measurements (each being the mean of two measurements) is $\sqrt{\frac{2}{n}}\,a$.

[For p. e. of a reading $= a$
$\therefore$ p. e. of a combined measurement $= a$
and p. e. of mean of n combined measurements $= \frac{a}{\sqrt{n}}$, etc.]

Ex. 14. A theodolite is furnished with n reading microscopes, all of the same precision. A graduation-mark on the limb is read on m times with a single microscope, giving the p. e. of a single reading to be r_1. The telescope is then pointed at an object m times, and the p. e. of the mean of the microscope readings is found to be r_2. Show that the p. e. of a pointing is

$$\sqrt{r_2^2 - \frac{r_1^2}{n}}$$

[p. e. of reading (mean of verniers) with n microscopes $= \frac{r_1}{\sqrt{n}}$

Total error = error of reading + error of pointing

$\therefore r_2^2 = \frac{r_1^2}{n} +$ (p. e. of ptg.)^2, etc.]

Ex. 15. If $\mu_1 b_1$, $\mu_2 b_2$ are the m. s. e. of the base measurements, and $\mu_3\lambda$ the m. s. e. of the ratio λ, given by the triangulation, of a base b_2 to a base b_1, show that the m. s. e. of the discrepancy between the computed and measured values of b_2 is $b_2\sqrt{[\mu^2]}$.

[Discrepancy $= b_2 - b_1\lambda = l$ suppose

$$\therefore \quad db_2 - b_1 d\lambda - \lambda db_1 = dl$$

and $\mu_2^2 b_2^2 + b_1^2(\mu_3\lambda)^2 + \lambda^2(\mu_1 b_1)^2 = \mu^2$,

or $b_2^2(\mu_2^2 + \mu_3^2 + \mu_1^2) = \mu^2$]

Ex. 16. At the time t_1 the correction to a chronometer was $a_1^s \pm r_1$, and at the time t_2 it was $a_2^s \pm r_2$; show that the p. e. of the rate of the chronometer is $\dfrac{\sqrt{r_1^2 + r_2^2}}{t_2 - t_1}$ and find the p. e. of the correction to the chronometer at an interpolated time t'.

$$\left[\text{Correction} = a_1 + \frac{a_2 - a_1}{t_2 - t_1}(t' - t_1) \text{ at time } t',\right.$$

$$\left.\therefore \text{p. e.} = \frac{\sqrt{(t_2 - t')^2 r_1^2 + (t' - t_1)^2 r_2^2}}{(t_2 - t_1)}\right]$$

Ex. 17. Given

$$x \cos \alpha = l_1 \pm r_1$$
$$x \sin \alpha = l_2 \pm r_2$$

find p. e. of x and of α.

$$\left[dx = \frac{\partial x}{\partial l_1} dl_1 + \frac{\partial x}{\partial l_2} dl_2\right.$$

$$= \frac{l_1}{\sqrt{l_1^2 + l_2^2}} dl_1 + \frac{l_2}{\sqrt{l_1^2 + l_2^2}} dl_2$$

$$\left.\therefore \text{p. e. of } x = \sqrt{\frac{l_1^2 r_1^2 + l_2^2 r_2^2}{l_1^2 + l_2^2}}. \quad \text{Similarly p. e. of } \alpha = \frac{\sqrt{l_2^2 r_1^2 + l_1^2 r_2^2}}{l_1^2 + l_2^2}\right]$$

Ex. 18. Given on a line measure the p. e. of a distance OA measured from O to be r_1, and of OB, also measured from O, to be r_2; find the p. e. of OD when D is the middle point of AB.

[$OD = \frac{1}{2}(OA + OB)$

$$\therefore r = \tfrac{1}{2}\sqrt{r_1^2 + r_2^2}$$

If $r_1 = r_2 = 1^\mu$, for example, then $r = 0^\mu.8$, when $\mu =$ one micron.

It may at first sight appear paradoxical that the p. e. of the computed quantity may be smaller than the p. e. of the measured. It is evident, however, that the error of OD is one-half the sum of the errors of OA and OB. If the signs of the errors are alike the error of OD is *never* greater than the larger of the errors; if the signs are different it is *always* less.]

Ex. 19. Given the p. e. of x to be r; find the p. e. of log x.

$$\left[d \log x = \frac{\text{mod.}}{x}\, dx \right.$$

$$\left. \therefore \text{p. e. } \log x = \frac{\text{mod.}}{x}\, r \right]$$

Ex. 20. In the measurement of the Massachusetts base line, consisting of 2165 boxes, the p. e. of a box, as derived from comparisons with the standard meter, was $\pm\, 0^m.0000055$, the p. e. from instability of microscopes in measuring a box was $0^m.000127$, and the p. e. of the base from temperature corrections was $0^m.0332$. Show that the p. e. of the base arising from these independent causes combined is $0^m.0358$.

$$[\text{p. e.} = \sqrt{(2165 \times 0^m.0000055)^2 + (0^m.000127\ \sqrt{2165})^2 + (0^m.0332)^2}$$
$$= \pm\, 0^m.0358]$$

Ex. 21. Given the length of the Massachusetts base to be $17326^m.3763 \pm 0^m.0358$; show that the corresponding value of the p. e. of its logarithm is 8.973 in units of the seventh place of decimals.

$$[\ \log (b \pm 0.0358) = \log b \pm \frac{\text{mod.}}{b}\, (0.0358) \qquad \text{See Art. 7.}$$

log mod.	9.6377843
log 0.0358	8.5538830
	8.1916673
log b	4.2387077
0.0000008973	3.9529596]

Ex. 22. The m. s. e. of the log. of a number N in units of the seventh decimal place is $\pm$ 10.6; find the ratio of the m. s. e. to the number.

$$[\ \log (N + v) = \log N + \frac{\text{mod.}}{N}\, v$$

$$\mu^2_{\log (N+v)} = \left(\frac{\text{mod.}}{N}\right)^2 \mu^2$$

$$\therefore\ \frac{\text{mod.}}{N}\,\mu = 10.6 \div 10^7$$

$$\text{and} \qquad \left.\frac{\mu}{N} = \frac{1}{410000}\right]$$

67. Examples of Weighting.

Ex. 1. The weights of the independently measured angles BAC, CAD, DAE are 3, 3, 1 respectively; find weight of the sum-angle BAE.

$$\left[\frac{1}{\text{wt.}} = \frac{1}{3} + \frac{1}{3} + \frac{1}{1}, \quad \therefore\ \text{wt.} = 0.6\right]$$

Ex. 2. If $X = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, and $p_1, p_2, \ldots p_n$ are the weights of $x_1, x_2, \ldots x_n$, and P the weight of X, show that

$$\frac{1}{P} = \left[\frac{aa}{p}\right]$$

Ex. 3. Prof. Hall found, from observations of the satellites of Mars, that from Deimos, Mass of Mars $= \dfrac{1}{3095313 \pm 3485}$, and from Phobos, Mass of Mars $= \dfrac{1}{3078456 \pm 10104}$, the mass being expressed in the common unit. Show that, taking the weighted mean, we have approximately

$$\text{Mass of Mars} = \frac{1}{3093500 \pm 3295}$$

Ex. 4. On a graduated bar the space $0 - 1^m$ is measured and found to be $1m.$ with a weight 1, and the space $0 - 2^m$ is measured and found to be $2m.$ with a weight 2; required the value of the space $1^m - 2^m$ and its weight P.

[Space $1^m - 2^m = 1m.$ It makes no difference what the weights are so far as the value of the space is concerned.

To find P. $(1^m - 2^m) = (0 - 2^m) - (0 - 1^m)$

$$\therefore \frac{1}{P} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \text{ and } P = \frac{2}{3}\ ]$$

Ex. 5. Given the weight of $x = p$, show that

$$\text{weight of } \log x = \frac{x^2}{(\text{mod.})^2} p$$

Ex. 6. If $x = \dfrac{y}{c}$ and the weight of y is p, then

$$\text{weight of } x = c^2 p$$

Ex. 7. Given the results for difference of longitude, Washington and Key West,

		m. s. s.
1873,	Dec. 24,	19 01.42 ± 0.044
	Dec. 26,	1.37 ± .037
	Dec. 30,	1.38 ± .036
	Dec. 31,	1.45 ± .036
1874,	Jan. 9,	1.60 ± .046
	Jan. 10,	1.55 ± .045
	Jan. 11,	19 01.57 ± 0.047

show that

	m. s. s.
weighted mean	= 19 01.460 ± 0.016
weighted mean of first four nights	= 19 01.404 ± 0.019
weighted mean of last three nights	= 19 01.573 ± 0.027

and from the last two results check the first.

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Ex. 8. In the triangulation connecting the Kent Id. Base, Md., and the Craney Id. Base, Va., the length of the line of junction computed from

$$\begin{aligned}\text{Kent Id. Base} &= \overset{m.}{26758.432} \pm \overset{m.}{0.38}\\ \text{Craney Id. Base} &= 26758.176 \pm 0.43\end{aligned}$$

Show that

(1) Discrepancy of computed values $= \overset{m.}{0.256} \pm \overset{m.}{0.57}$
(2) Most prob. length of junction line $= 26758.32 \pm 0.028$

Ex. 9. In latitude work with the zenith telescope, if n north stars are combined with s south stars, giving ns pairs, to find the weight of the combination, that of an ordinary pair, one north and one south, being unity.

[Let $\mu =$ the m. s. e. of an observation of one north star or of one south star.

Then, as though combining the mean of n north stars with the mean of s south stars, the wt. p of the combination is

$$\frac{1}{p} = \frac{\mu^2}{n} + \frac{\mu^2}{s}$$

But

$$1 = \frac{\mu^2}{1} + \frac{\mu^2}{1}$$

$$\therefore\ p = \frac{2ns}{n+s}\ ]$$

"The combination of more than two stars gave some trouble. In one case there were 3 north and 4 south, which would give 12 pairs, but with a weight of $2\frac{3\times 4}{3+4}$ only. In this and all similar cases I treated the whole combination as one pair; that is, I inserted in the blank provided the half-sum of the mean of the declinations of north stars and of the mean of the declinations of south stars, and gave the result a higher weight. This is the only logical method." (Safford, *Report, Chief of Engineers U. S. A.*, 1879, p. 1987.)

For a series of examples by Airy on the weights to be given to the separate results for terrestrial longitude determined by the observations of transits of the moon and fixed stars, see *Mem. Roy. Astron. Soc.*, vol. xix.

Ex. 10. If a close zenith star is observed with a zenith telescope first as a north star, and immediately after as a south star, show that the weight of the resulting latitude is less than that found from observing an ordinary pair.

Ex. 11. In the triangulation of Lake Ontario the angle Walworth-Palmyra-Sodus was measured as follows:

In 1875, with theodolite P. and M. No. 1,

$74^\circ\ 25'\ 05''.429 \pm 0''.29$, mean of 16 results;

In 1877, with theodolite T. and S. No. 3,

$74^\circ\ 25'\ 04''.611 \pm 0''.22$, mean of 24 results;

required the most probable value of the angle and its probable error.

[With first theodolite p. e. of a single obs. $= 0''.29 \sqrt{16} = 1''.16$

With second theodolite p. e. of a single obs. $= 0''.22 \sqrt{24} = 1''.08$

Let a single result with the first theodolite be taken as unit of weight, then mean of 16 results has weight 16.

Let a single result with the second theodolite have a weight p, referred to the same unit as the first, then mean of 24 results has weight $24\,p$. The value of p is found from the relation

$$\frac{p}{1} = \left(\frac{1.16}{1.08}\right)^2$$

Also

$$\text{most prob. value of angle} = 74^\circ\ 25'\ 04'' + \frac{1''.429 \times 16 + 0''.611 \times 24\,p}{16 + 24\,p}$$

and

$$\text{weight of this value} = 16 + 24\,p\ ]$$

Note.—If, instead of being two measurements of the same angle, the above were the measurements of two angles side by side, then

$$\text{total angle} = 148^\circ\ 50'\ 10''.040$$

because, no matter how much better one is measured than the other, we can do nothing but take the sum of the two values. The weight P of the result would be found from

$$\frac{1}{P} = \frac{1}{16} + \frac{1}{24\,p}$$

Ex. 12. An angle is measured n times with a repeating theodolite, and also n times with a non-repeating theodolite, the precision of a single reading and of a single pointing being the same in both cases; compare the weights of the results.

[μ_1, μ_2 the m. s. e. of a single pointing and of a single reading.

With a non-repeating theodolite each measurement of the angle contains

$$(\text{pointing} + \text{reading}) - (\text{pointing} + \text{reading})$$

$$\therefore\ (\text{m. s. e.})^2 \text{ of one measurement} = 2\mu_1^2 + 2\mu_2^2$$

$$\text{and } (\text{m. s. e.})^2 \text{ of mean of } n \text{ measurements} = \frac{1}{n}(2\mu_1^2 + 2\mu_2^2).$$

With a repeating theodolite the successive measurements of the angle are

$$\begin{aligned}(\text{pointing} + \text{reading}) &- \text{pointing}\\ \text{pointing} &- \text{pointing}\\ \cdot\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\\ \text{pointing} &- (\text{pointing} + \text{reading})\end{aligned}$$

$$\therefore\ (\text{m. s. e.})^2 \text{ of } n \text{ times the angle} = 2n\mu_1^2 + 2\mu_2^2$$

$$\text{and } (\text{m. s. e.})^2 \text{ of the angle} = \frac{1}{n^2}(2n\mu_1^2 + 2\mu_2^2)$$

If, then, p_1, p_2 denote the weights of an angle resulting from n reiterations or from n repetitions,

$$p_1 : p_2 = 1 + \frac{\mu_2^2}{n\mu_1^2} : 1 + \frac{\mu_2^2}{\mu_1^2}$$

and hence it would seem that the method of repetition is to be preferred to the method of reiteration. This advantage is so much less, the smaller $\frac{\mu_2^2}{\mu_1^2}$ is; that is, the more the precision of the circle reading increases in proportion to the precision of the pointing.

This result is contradicted by experience—so much so that in all of the leading surveys repeating theodolites are no longer used in primary work. Where, then, is the fault? Is the theory of least squares false? By no means. We have only another example of a point which occurs over and over again, and which is so apt to be overlooked. (See Arts. 53, 54.)

The result obtained is true on the hypothesis that only accidental errors enter. We have assumed a perfect instrument. But the instrument-maker cannot give what the geometer demands. From various mechanical reasons the systematic error in a repeating theodolite increases with the number of observations, whereas in the reiterating theodolite it disappears. This systematic error, in whatever way it arises, causes the trouble. It is useless to discuss accidental errors until it is out of the way; and as no means have yet been devised of getting rid of it, the instrument itself has been abandoned.

Cf. Struve, *Arc du Méridien*, vol. i.; Louis Cruls, *Discussion sur les Méthodes de repetition et reiteration*, Gand, 1875; Herschel, *Outlines of Astronomy*, Art. 188; *Coast Survey Report*, 1876, App. 20.]

Ex. 13. An angle is measured with two repeating theodolites. With the first are made n_1 repetitions, μ_1, μ_1' being the m. s. e. of a pointing and of a reading; with the second are made n_2 repetitions, μ_2, μ_2' being the m. s. e. of a reading and of a pointing. Show that if n_1, n_2 are large the weights of the results are as

$$\frac{n_1}{\mu_1^2} : \frac{n_2}{\mu_2^2}$$

NOTE I.

ON THE WEIGHTING OF OBSERVATIONS.

68. When the sources of error are of such kinds that, so far as we know, they cannot be separated, the m. s. e. and consequent weight are found as described in the preceding sections. The weight has been defined as a number representing the relative goodness of an observation, and as a

number inversely proportional to the square of the m. s. e. of an observation. If in a series of observations the conditions required for the determination of the law of error could be strictly fulfilled, these two statements would lead to the same result. In actual cases, however, this is only approximately true. Thus two separate determinations of a millimeter space, made in the same way, gave

$$\overset{\mu}{1000}.1 \pm \overset{\mu}{0}.40, \text{ mean of 20 readings.}$$
$$1000.3 \pm 0.33, \text{ mean of 30 readings.}$$

To find the weighted mean of these two sets of measurements we may proceed in two ways. The number of results in the first measurement is 20, and the number in the second is 30. Hence, taking the weights proportional to the number of results, the mean

$$= 1000 + \frac{20 \times 0.1 + 30 \times .3}{20 + 30} = 1000.220$$

Again, since the p. e. of the measurements are 0.40 and 0.33, their weights are as $\frac{1}{40^2}$ to $\frac{1}{33^2}$, that is, as 1089 to 1600, and the resulting weighted mean is 1000.219, agreeing, within the limits of the p. e., with the other value.

In this example the two methods agree as nearly as could be expected from the small number of observations. But it is not always so. Some "run of luck," or balancing of error, or constant conditions might have made the observations of one set fall very closely together, in which case the weight as found from the p. e. would have been very large, while varying conditions might have caused wide ranges, giving a small weight. A great deal, therefore, depends on the judgment of the computer in deciding what weight is to be given, it being constantly kept in mind that the strict formulas which are correct in an ideal case must not be pressed too far in practice. Thus in the second set of observations above the first three results were 999.8,

999.8, 999.8. The p. e. computed from these would be zero, and the consequent weight infinite. But no one will doubt but that the mean of the 30 results is more reliable than the mean of these three results.

69. **An Approximate Method of Weighting.**—A long-continued series of observations will show the kind of work an instrument is capable of doing under favorable conditions; and if work is done only when the conditions are favorable, the p. e. derived from a certain number of results will generally fall within limits that can be assigned *à priori*. For example, with the Lake Survey primary theodolites, which read to single seconds, the tenths being estimated, the work of several seasons showed that the p. e. of the mean of from 16 to 20 results of the value of a horizontal angle, each result being the mean of a reading with telescope direct and of a reading with telescope reverse, need not be expected to be greater than $0''.3$. If, therefore, after having measured a series of angles in a triangulation net with these instruments, the p. e. all fell within $\pm 0''.3$, it was considered sufficiently accurate to assign to each angle the same weight.

The objection to this is that "an instrument which has a large periodic error may, if properly used, give as good results as if it had none; but the discrepancies between its combined results for an angle and their mean may be large, thus giving an apparently large probable error to the mean. Moreover, a given number of results over short lines, or lines over which the distant signals are habitually steady when seen in the telescope, will give a resulting value for the angle of much greater weight than the same number of combined results between two stations which are habitually unsteady." *

The same method of weighting was employed by the Northern Boundary Commission in their latitude work. "The standard number of observations [for a latitude determination] was finally fixed at about 60, it being found

* *Professional Papers of the Corps of Engineers U. S. A.*, No. 24, p. 354.

that with the 32-in. instrument 60 observations would give a mean result of which the p. e. would be about 4 feet." *

70. **Weighting when Constant Error is Present.**—The preceding leads us to the case where the error of observation can be separated into two parts, one of which is due to accidental causes and the other to causes which are constant throughout the observations. The total error e would, therefore, be of the form

$$e = l_1 + l_2$$

This case has been discussed already in general terms in Art. 53 in explaining the well-known fact that an increase in the number of observations with a given instrument does not lead to a corresponding increase of accuracy in the result obtained.

Let

μ_1 = the m. s. e. of the observation arising from the accidental causes,

μ_2 = the error peculiar to the observation arising from the constant causes.

Then μ_1, μ_2 being independent, the total m. s. e. μ of observation may be assumed

$$\mu^2 = \mu_1^2 + \mu_2^2$$

If n observations have been made we shall have for the m. s. e. μ_o of their mean, since μ_2 is constant,

$$\mu_o^2 = \frac{\mu_1^2}{n} + \mu_2^2$$

It is evident that when n is large μ_2^2 becomes the important term, and that in any case the value of μ_o and consequent weight can be but little improved by increasing the number of observations.

For the purpose of finding the value of the m. s. e. arising from the constant sources of error a special series of observations is in general necessary. After this series has

* *Report, Survey of the Northern Boundary*, p. 86.

been made the value of μ_2 found from it can be applied in the determination of the value of μ_0 in any other series made under like conditions.

For illustration let us consider a latitude determination with the zenith telescope. It is well known that with this instrument a latitude result found from two observations of a single star either north or south of the zenith is inferior to one found from a combination of a north and a south star. This arises, not from any difference in the mode of observation, but from the errors in declination as given in the star catalogue being cumulative in the one case and balancing in the long run in the other.

The zenith-distance, ζ, of each star being observed, the half-difference of zenith-distances for each pair may be computed, and each of these computed values may be considered an observed value. The values of the declinations δ are taken from a catalogue of stars. The errors of δ are, therefore, independent of those of ζ, and are constant for the same pair of stars. The latitude φ from one pair is given by

$$\varphi = \tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(\zeta' - \zeta)$$

Let

$\mu_\zeta =$ the m. s. e. of $\frac{1}{2}(\zeta' - \zeta)$ for one observation of one pair,
$\mu_\delta =$ the m. s. e. of $\frac{1}{2}(\delta' + \delta)$ for this pair,
$\mu_\phi =$ the m. s. e. of the resulting latitude φ from one pair,

then for a single observation of this pair

$$\mu_\phi^2 = \mu_\delta^2 + \mu_\zeta^2$$

and for n observations of this pair

$$\mu_\phi^2 = \mu_\delta^2 + \frac{\mu_\zeta^2}{n}$$

The quantity μ_ζ will be found from repeated observations of the same pair of stars, as the error in declination will not influence the result. A better value will, of course, be obtained from several pairs than from a single pair. Let, then, many pairs of stars be observed night after night for

a considerable period. Collect into groups the latitudes resulting from the observed values of each separate pair. Let n_1, n_2, . . . n_m be the number of results in the several groups, the number in any group being at least two. Form the residuals for each group and compute the m. s. e. in the usual way. We have:

No. of Night.	First Pair.		Second Pair.		. . .
	Results.	v	Results.	v	. . .
1	φ_1'	v_1'	φ_2'	v_2'	. . .
2	φ_1''	v_1''	φ_2''	v_2''	. . .
3	φ_1'''	v_1'''	φ_2'''	v_2'''	. . .
. .	. .	. .	. .	. .	. . .
Means	φ_1		φ_2		. . .

Now, assuming that the m. s. e. of observation of each pair is the same,

$$\mu_\zeta^2 = \frac{[v_1 v_1]}{n_1 - 1}$$

$$\mu_\zeta^2 = \frac{[v_2 v_2]}{n_2 - 1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

If, then, n is the total number of results, and m the number of groups, by adding the above equations there results

$$\mu_\zeta^2 = \frac{[vv]}{n - m}$$

In finding μ_ϕ we assume that though errors of declination are constant for each star, still for a latitude found from many pairs in the same catalogue these errors may be

regarded as accidental. Let, then, many different pairs of stars be observed on each of n nights at m' places, no star being observed at more than one place. Collect the means of the single results of each separate pair and form the residuals v' for each place, taking the differences between these means considered as single results and their mean for that place. Then, reasoning as above, the m. s. e. of a latitude resulting from n observations on a single pair of stars is

$$\mu_\phi = \sqrt{\frac{[v'v']}{n' - m'}}$$

where n' is the number of different pairs of stars observed and m' is the number of places occupied.

Now, μ_δ is found from

$$\mu_\delta^2 = \mu_\phi^2 - \frac{\mu_\varsigma^2}{n}$$

and is, therefore, known for the star catalogue used. This value may be taken in future work in finding μ_ϕ from

$$\mu_\phi^2 = \mu_\delta^2 + \frac{\mu_\varsigma^2}{n}$$

and the consequent combining weight of φ will be as

$$\frac{1}{\mu_\phi^2}$$

An example of a similar kind is afforded in finding the weights of the angles measured with a theodolite in a triangulation where more rigid values are required than would be found by Art. 69. The actual error of a measured value of an angle arises from two main sources, errors of graduation and errors of observation The former are constant for different parts of the limb read on, and correspond to the declination errors above, while the latter are incapable of classification, and are, therefore, assumed to be accidental. The periodic errors of graduation are supposed to have

been eliminated by proper shiftings of the circle. The resultant m. s. e. μ of a single measurement is found from

$$\mu^2 = \mu_1^2 + \mu_2^2$$

and the m. s. e. μ_0 of the mean of n measurements made on the same part of the limb from

$$\mu_0^2 = \mu_1^2 + \frac{\mu_2^2}{n}$$

where μ_1, μ_2 are the m. s. e. of graduation and observation respectively. The method of treating this problem is quite similar to that of the preceding; μ_2 is found by reading the same graduation-mark on the limb many times, and μ_0 by reading the angle between two fixed signals many times, the limb being changed after each reading. Thence μ_1 is known for the instrument in question, and the combining weights of angles measured with this instrument are at once found.

The foregoing leads to another important practical point in the measurement of angles. If the weight of a single observation is unity, then the weight of the mean of n observations made with the limb in one position is

$$p = \frac{\mu_1^2 + \mu_2^2}{\mu_1^2 + \frac{\mu_2^2}{n}}$$

Experience has shown that we may safely assume

$$\mu_1 = \mu_2$$

and therefore it follows that

$$p = \frac{2n}{n + 1}$$

Hence, no matter how many observations we make in one position of the limb, we never reach the precision of the mean of two observations made with the limb in different positions.

It might fairly be inferred that the limb should be shifted after each single reading of an angle, and the rea-

sons for not doing so are to guard against mistakes in reading and to eliminate twist of station, as explained elsewhere. (Art. 54.)

71. **Assignment of Weight Arbitrarily.**—So far we have deduced the combining weights from the observed values themselves, or from them in connection with a special series of observations. But this may not always be the best way of finding the weights. The observations may not be our only source of information, and, indeed, not the most reliable source. If, for example, some phenomenon has been observed by many persons in different parts of the country, and the observations are sent to one place for comparison and reduction, it would not be proper for the computer to deduce a weight for each series from the observations themselves independent of other sources of information he might have. Some of the most inexperienced observers with the poorest instruments might have apparently better results than the most experienced with good instruments. In such a case the computer must exercise his own judgment in classing the observations. He should consider the experience of the observer, his previous record for accurate work, the kind of instrument used, the conditions, and the observer's record of what he saw—whether it is clear and precise or hazy in its statements. An arbitrary scale of weights may then be constructed, and to each set of observations be assigned a weight from this scale according to the computer's estimate of its value. No two computers would be likely to assign precisely the same weights, but if done by one of experience and good judgment the result obtained from weighting in this way will undoubtedly be of more value than that found by the strict application of the formulas of least squares.

The point is simply this. The class of observations considered may be expected to contain systematic errors which cannot be determined, and is therefore not capable of being treated by the method of least squares. As we have no direct means of eliminating this kind of error, we must do so

indirectly as best we can, and that is what the system of weighting mentioned seeks to accomplish.

An example will be found in the discussion of the *Telescopic Observations of the Transit of Mercury*, May 5–6, 1878, Washington, 1879, where, of 109 observations sent in, to only 18 was the highest weight assigned. Prof. Eastman, under whose direction they were reduced, says: ". . . Several instances may be found where small weight is given to observations that apparently agree well with those to which the highest weight is assigned, but in most cases the observer's remarks indicate the uncertain character of the observation."

72. **Combination of Good and Inferior Work.**—It is strictly in accordance with the idea of weight that if we have two results of very different degrees of accuracy, a result better on the whole than either may be found by combining both with their proper weights. But the proper weights may be difficult to find. On this account it depends on circumstances whether it is advisable to reduce a set of observations poorly made, in order to combine them with a well-made set. If the quantity is available for observing again it might not cost any more to do this than to reduce the poor observations. Even if it did the result would be more satisfactory. The committee of the Royal Society of England which was appointed to examine Col. Lambton's geodetic work in India reported that "Col. Lambton's surveys, though executed with the greatest care and ability, were carried on under serious difficulties, and at a time when instrumental appliances were far less complete than at present. There is no doubt that at the present time the surveys admit of being improved in every part. The standards of length are better ascertained than formerly, and all uncertainty on the unit of measure may be removed. The base-measuring apparatus can be improved. The instruments for horizontal angles used by Col. Lambton were inferior to those now in use. . . . The committee express the strong hope that the whole of Col.

Lambton's survey may be repeated with the best modern appliances." *

73. **The Weight a Function of our Knowledge.**—If a quantity is not available for observing again, as, for example, some transient phenomenon, all of the material on hand must be used, and the best weights possible assigned to the separate values in order to combine them. The point is that where systematic or constant error has not been eliminated the weight to be assigned is a function of the state of our knowledge—is, in fact, a matter of individual judgment.

This is brought out very fully in the methods used in combining the older star catalogues with the more modern ones. Thus Safford (*Catalogue of Mean Declinations of 2018 Stars*, Washington, 1879) says: "In computing positions I have generally employed Argelander's rule giving to a modern determination from

1 observation a weight $\frac{1}{2}$,
2 observations a weight $\frac{3}{4}$,
3 to 8 observations a weight 1,
9 or more observations a weight $1\frac{1}{2}$ or 2.

Argelander generally gives Piazzi a weight equal to unity; the value $\frac{1}{2}$ is much nearer the truth; in general he assigns rather a larger relative weight to the older and poorer observations than they deserve. But this is mostly compensated for by the number of determinations."

The weight of a quantity being a function of our knowledge may have assigned to it a certain value at one time and another value at another time when our knowledge of it has increased. Thus in the Fond du Lac (Wis.) base, measured in 1872 with the Bache-Würdemann compensating apparatus, a portion was measured seven times. The results differed widely, far beyond what was expected with the apparatus. No reason could be assigned at the time for the discordances. At this stage, then, one would have been

* *G. T. Survey of India*, vol. ii. p. 70.

justified in assigning a small weight to the value of the base.

The Keweenaw base was next measured with the same apparatus, and the same trouble came in. Next the Sandy Creek base and then the Buffalo base were measured. During all this time (four years) material had been accumulating for the explanation of the behavior of the apparatus. When the law of its behavior was discovered it was found that good work not only could be done but had been done with it.

Hence the systematic error being got rid of, one would be justified in increasing the weights of the bases measured with this apparatus in comparison with bases measured with an apparatus of a different kind. Had the later work not been done the Fond du Lac base would still have had assigned to it the low weight.

Take another instance. Sir G. B. Airy, in 1847, says of the Mason and Dixon arc (*Encyc. Metrop.*, p. 209): "The results of this measure must, we think, be received as equal in authority to those of any other measure." This may have been true when written; but Mr. Schott, in 1877, in his note on the determination of the figure of the earth from American sources, says of this same arc (*U. S. C. S. Report*, 1877, p. 95): "It is, therefore, only owing to the increased perfection of instrumental means and methods that we now dismiss from further consideration the first measured North American arc, which, moreover, is now superseded by the present measures."

As a third illustration we may consider the weights to be assigned to a system of differences of longitudes in which the connections of the stations occupied are interlaced as in a triangulation net, and the whole system is to be adjusted so as to remove existing contradictions.

If the longitude work has been carried out on one plan, with instruments and observers of about the same quality, then the m. s. e. of each determination may be computed from the measures of the separate nights, and in the adjust-

ment the weights may be taken inversely as the squares of these m. s. e.

But if this has not been done, if in the older work instruments, observers, and methods were poorer than later and the two have to be combined in the adjustment, the computer must estimate as best he can their relative weights. Thus in a system in Germany, France, and Austria reduced by Dr. Albrecht* the observations were made between the years 1863 and 1876. The methods of observation had been much improved in this interval. In assigning the relative weights a scale of weights was first formed from a consideration of all the knowledge on hand, taking the march of improvement from year to year into account, and the separate determinations placed in one or other of these classes. Thus, for example,

Weight 1—No change of observers; few observations; non-adjustment of electric current;
Weight 2—No change of observers; usual variety of observations; non-adjustment of electric current;
Weight 3—Change of observers; usual variety of observations; non-adjustment of electric current,

and so on.

Similarly Dr. Bruhns in *Verhandlungen der europäischen Gradmessung*, 1880. See also *Coast Survey Report*, 1880, Appendix 6.

74. **General Remarks.**—The subject of the weighting of observations is confessedly a difficult one. In general it may be affirmed that the less experienced a computer is the more closely he will adhere to the rigorous formulas without considering whether systematic errors enter or not. As he adds to his experience he will consider outside evidence as well as the evidence afforded by the observations themselves. This will be specially true if he has any practical knowledge of how observations are made. Indeed, it is doubtful if a computer can apply the principles of

* *Astronomische Nachrichten*, 2132.

least squares properly unless he is at least an average observer.

Great caution, however, is necessary in assigning weights, because it is sometimes possible so to choose them as to make observations tell anything desired. They should always be chosen from a consideration of all the evidence on hand, and may be changed as additional evidence is presented, so that a result is never final, but is ever open for improvement. The original records of the observations and the methods of reduction are quite as desirable, if indeed not more so, than the results deduced. The best plan, therefore, is to publish all of the data along with the reduction, when the reader, if he wishes, can make a reduction for himself. He can then form a more intelligent opinion of the computer's skill and judgment, and also of the value of the work.

NOTE II.

ON THE REJECTION OF OBSERVATIONS.

75. There is nothing in the whole theory of errors more perplexing than the question of what shall be done with an observation of a series which differs widely from the others. In making a series of observations the observer is given full power. He can vary the arrangements, choose his own time for working, reject any result or set of results; he can do anything, in fact, that in his best judgment will tend to give the best value of the observed quantity. But when he has finished observing and goes to computing, has he the same power? Can he alter, reject, manipulate in such a way as in his best judgment will give a result of maximum probability? As observer he was supreme; as computer is he supreme, or only in leading-strings? Various answers

may be given to this question, as we look at it from one point of view or another.

In the hypothetical case on which the exponential law of error was founded there were no discontinuous observations taken into account. There we contemplated not only observations made with the best instruments and by the most experienced observers, but observations of all grades, from this highest grade down to those made with the poorest instruments and by the most ignorant and careless observers conceivable. It is only in this way that errors continuous all the way from $+\infty$ to $-\infty$ could arise. In the cases occurring in ordinary work we confine our attention to one section of the observations only—that made with the good instrument and by the skilful observer. This, to be sure, is the most important, and, as shown in Art. 30, the result following from it differs ordinarily but little from that found in the ideal case. But we are naturally confronted with difficulty when we try to deal with a very incomplete series. Extra assumptions must be made, and it is not to be wondered at that no solution yet offered is regarded as entirely satisfactory.

76. A common summary method of disposing of the subject is contained in the following statement: "The weights [of the angles] would have been materially increased in many instances by rejecting what would appear *bad* observations; but the rule has been never to reject any unless the observer has made a remark to the effect that it ought to be rejected." This statement, however, does not cover the whole ground. Those who reason in this way make a distinction between mistakes and errors of observation. Mistakes are rejected. But the great difficulty is to tell just where mistakes end and errors begin.

Given a set of measures involving discordances unlooked for, and which the observer's remarks do not cover, how shall we proceed? Two views may be taken. In the first place, the computer, from a consideration of the measures themselves and from all other evidence bearing on them

that he can discover, may make a distinction between measures and mistakes which will do for the set before him. With observations of another kind he might have a different mode of procedure. Another computer might have different rules altogether, precisely as in the case of weighting as explained in Arts. 71–74.

To put the discrepant values with the other values, and take the arithmetic mean of all, would give a result considerably different from what would be found by omitting them. It would take a great many good observations to balance the effect of a single widely discrepant one. It is, then, for the computer to judge whether this discrepant observation shall have the same weight as the others or a different weight.

It may happen, indeed, that the discrepancy is so evidently a "natural mistake" that it may be corrected without a doubt from the evidence furnished by the other observations, and the discrepant observation changed so that it may be treated as a good one. Thus an angle may be read 5′ or 10′ wrong, or a micrometer screw may be read 5 or 10 revolutions out of the way, as shown by the rest of the observations, and the like.

Or the observations may be arranged in well-defined groups, and if the computer finds that he cannot account for the presence of unusual discrepancies in a certain group, he may decide to reject the whole group. For example, in longitude work it may happen that one of the time stars may give a clock correction differing, say, one second of time from that given by fifteen or twenty others observed on the same night. Instead of rejecting the single star it would perhaps be better to reject the whole night's work. If necessary, an extra set of observations may be made to fill the blank. By rejecting a group no hidden law can be slighted, for if any exists it will continue to reappear in further observations, and finally to reveal itself.

77. In the second place, the computer, instead of trusting to his judgment, may call in the aid of the calculus of proba-

bilities and seek to establish a test or criterion for the rejection of observations which will serve for all kinds of observations. Of the criterions which have been proposed the earliest is due to Prof. Peirce. It is as follows: "Observations should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making so many and no more abnormal observations." A proof by Dr. Gould will be found in the *U. S. Coast Survey Report*, 1854, pp. 131, 132. It is founded on the assumption of the Gaussian law of error.

Another criterion "for the rejection of one doubtful observation" is given by Chauvenet in his *Astronomy*, vol. ii. p. 565. "We have seen that the function [Art. 30]

$$\frac{2h}{\sqrt{\pi}}\int_0^{\rho\frac{a}{r}} e^{-h^2\Delta^2}\,d\Delta$$

represents in general the number of errors less than a which may be expected to occur in any extended series of observations when the whole number of observations is taken as unity, r being the p. e. of an observation. If this be multiplied by the number of observations n, we shall have the actual number of errors less than a; and hence the quantity

$$n - n\,\Theta(t) = n\{1 - \Theta(t)\}$$

expresses the number of errors to be expected greater than the limit a. But if this quantity is less than $\frac{1}{2}$ it will follow that an error of the magnitude a will have a greater probability against it than for it, and may, therefore, be rejected. The limit of rejection of *a single doubtful observation* is, therefore, obtained from the equation

$$\tfrac{1}{2} = n\{1 - \Theta(t)\}$$

or

$$\Theta(t) = \frac{2n - 1}{2n}\ \text{"}$$

A third criterion was proposed by Mr. Stone, Radcliffe observer at Oxford, Eng., in *Month. Not. Roy. Astron. Soc.*, 1868, 1873, in these terms: "I assume that a particular person, with definite instrumental means and under given circumstances, is likely to make, on an average, one mistake in the making and registering n observations of a given class. The probability, therefore, is that any record of his of this class of observations as a mistake is $\frac{1}{n}$. From the average discordances among the registered observations of this class we can find the p. e. of an observation in the usual way, and also the probability of an error greater than a given quantity, as C. Then if the probability in favor of a discordance as large as C is less than that of a mistake, or $\frac{1}{n}$, the discordant observation is rejected."

The least objectionable criterion based on mathematical principles may, I think, be developed from the principle laid down in Art. 50, where the maximum error was estimated at about five times the p. e. or three times the m. s. e. If, therefore, an observation differs from the general run of the series by more than this amount it should at least be bracketed and attention be called to it.

78. It may be stated that, as a general rule, criterions are apt to be most highly esteemed by those who look at the observations from a purely mathematical rather than from the practical observer's point of view. The latter is, without doubt, the true standpoint. Every observer will, consciously or unconsciously, construct a criterion suited to the sort of work he is engaged in. This criterion will not necessarily be founded altogether on mathematical formulas. Indeed, most likely it will not be. Nor does it follow that the criterion adopted in any special series is of universal application or will receive universal assent. In the process of weighting the observer will not assign weights always as the inverse square of the m. s. e. It is often better to assign them arbitrarily from a feeling founded on a general grasp of all the circumstances connected with the making of the observations. In like man-

ner, and on this same feeling, he will found his criterion for rejection. There is no uniform rule for weighting, neither is there one for rejection.

A criterion such as Stone's, for example, would be very useful in the work for which it was proposed, and in its proposer's hands would be of much value. But that any one without the insight given by long familiarity into the special kind of work from which this criterion arose could apply it properly is not to be expected. As already pointed out, in the case of weighting the only thing for the computer to do is to publish all of the observations, including those rejected, along with his reduction, when, if more light can at any time be thrown on them, a new reduction can be made. Take, for example, Bradley's observations as reduced by Bessel and Auwers (Art. 51). It cannot be too strongly insisted on that a result deduced from a series of observations is never to be looked on as final, but as ever open for improvement.

79. The difficulty in combining single observations lies in assigning to them their proper weights. We have assumed the arithmetic mean to give the most probable value. If the sources of error could be separated, so that to each single observation could be ascribed its proper weight, the resulting weighted mean would be nearer the truth than the direct arithmetic mean. We can, therefore, *conceive* of a better value than the arithmetic mean in certain cases.* This has been already pointed out in Art. 11.

We may, therefore, consider whether, when discrepant observations occur we may not get a more satisfactory result by ignoring the arithmetic mean altogether. Suppose, for example, that we had three observed values, 100, 60, 61, and that we had no means of getting any further observations. These values would seem to show that the true value was likely to be nearer 60 than 100. Just

*De Morgan (*Encyc. Metrop.*, "Theory of Probability," p. 456) suggested that the combining weights might be found from the observed values themselves, but he did not develop his plan, and it is apparently fruitless.

how much nearer is the question. To reject the observation 100 would be without reason, and to take the arithmetic mean of all three would ignore the evidence afforded by the observations themselves.

The observations being discontinuous, the question is removed from the theory of least squares, which presupposes continuity (see Art. 30), and must be treated by other methods. Laplace* discussed this problem long before Legendre and Gauss developed the method of least squares. The subsequent confusion in the introduction of criterions has arisen from trying to fasten to the method of least squares what is in reality a very different question.

When the observed values are discontinuous we cannot reasonably assume the value of the unknown sought to be a symmetrical function of them (see Art. 11). A plausible result has been given (Arts 11, 15) as that observed value which has as many observed values greater than it as it has less than it. Thus in the preceding example a good value to choose would be 61.

In discussing such problems, so long as not much greater plausibility can be assigned to one method of combination than to another, the question of convenience of computation comes in. In respect to this the propriety of selecting the middle term stands pre-eminent.

80. It is ever to be kept in mind that unexpected discrepancies in his results do not always prove to the observer that his work is bad, any more than a close agreement among them shows it to be good. When either occurs great caution is necessary. It is unsafe to have a rigid rule of any kind for sifting the observed values, not allowing the computer to make use of evidence outside of the observations. By following such rules we are apt to bar the way to discovery of new truths, or at least to hinder progress in that direction. See, for example, the history of the discovery of personal equation, *American Cyclopædia*, vol. xiii.

* *Mém. Acad. Paris*, vol. vi. p. 634.

Throughout this discussion it has been assumed that the observations have been reduced by the observer himself or by a computer who is at the same time a competent observer. A computer who is not an observer must of necessity employ the same criterion always; and in this case the criterion derived from Art. 50, as stated on page 135, is to be preferred.

The following memoirs may be consulted in addition to those already mentioned: Gergonne, *Annales de Math.*, vol. xii. pp. 181 seq.; Peirce in *Gould's Astron. Jour.*, vol. ii. pp. 161 seq.; Airy in do., vol. iv. pp 137, 138; Winlock in do., vol. iv. pp. 145–147; Stone, *Month. Not. Roy. Astron. Soc.*, vol. xxviii. pp. 165 seq., vol. xxxiv. pp. 9 seq.; Glaisher in do., vol. xxxiii. pp. 391 seq., also in *Mem. Roy. Astron. Soc.*, vol. xxxix. pp. 75 seq.

CHAPTER IV.

ADJUSTMENT OF INDIRECT OBSERVATIONS.

Determination of the Most Probable Values.

81. If direct measurements of a quantity have been made under the same circumstances, we have seen that the arithmetic mean of these measures gives the most probable value of the quantity. We now come to the case where the quantity measured is not the unknown required, but is a linear function of one or more unknowns whose values are to be found. This is the more general form, and its solution has been carried a certain distance in Art. 14. The point stopped at was the combining of observations of different weights. As by the aid of the law of error this can now be done, we proceed to finish the solution.

Let, as in Art. 14, the equations connecting a series of observed quantities $M_1, M_2 \ldots M_n$, n in number, and the independent unknowns $X, Y, \ldots$, n_i in number $(n > n_i)$, be

$$\begin{aligned} a_1X + b_1Y + \ldots - L_1 &= M_1 + v_1 \\ a_2X + b_2Y + \ldots - L_2 &= M_2 + v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \\ a_nX + b_nY + \ldots - L_n &= M_n + v_n \end{aligned} \qquad (1)$$

where $a_1, b_1, \ldots L_1, \ldots L_n$ are constants given by theory for each observation, and $v_1, v_2, \ldots v_n$ are the residual errors of observation.

In practice the labor of handling these equations will be much lessened by using an artifice we have several times already employed (see Art. 41). Let $X', Y', \ldots$ be close approximations to the value of $X, Y, \ldots$ found by ordinary elimination from a sufficient number of the equations, or by some other method, as by trial, for example, and put

$$X - X' = x, \quad Y - Y' = y, \ldots$$

where $x, y, \ldots$ are the corrections required to reduce the approximate values to the most probable values.

Then the observation equations reduce to

$$\begin{array}{l} a_1x + b_1y + \ldots - l_1 = v'_1 \\ a_2x + b_2y + \ldots - l_2 = v'_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + \ldots - l_n = v'_n \end{array} \qquad (2)$$

where

$$\begin{array}{l} -l_1 = a_1X' + b_1Y' + \ldots - L_1 - M_1 \\ -l_2 = a_2X' + b_2Y' + \ldots - L_2 - M_2 \\ \cdot \quad \cdot \quad \cdot \\ -l_n = a_nX' + b_nY' + \ldots - L_n - M_n \end{array}$$

and are, therefore, known quantities.

It is more convenient in practice to omit the residuals and write the observation equations in the form

$$\begin{array}{l} a_1x + b_1y + \ldots = l_1 \\ a_2x + b_2y + \ldots = l_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + \ldots = l_n \end{array} \qquad (3)$$

always keeping in mind that the strict form is as in (2).

The observation equations 3 may be written

$$\begin{array}{l} a_1x = l_1' \\ a_2x = l_2' \\ \cdot \quad \cdot \quad \cdot \\ a_nx = l_n' \end{array} \qquad (4)$$

if the values of $y, z, \ldots$ are supposed to be known.

Now, it has been shown in Art. 63 that the most probable value of x would be found from these equations by taking the weighted mean of the separate values $\frac{l_1'}{a_1}, \frac{l_2'}{a_2}, \ldots \frac{l_n'}{a_n}$. The weights of these values are in the same section shown to be as $a_1^2, a_2^2, \ldots a_n^2$. We have, therefore,

$$x = \frac{[al']}{[aa]}$$

or, by putting for l_1', l_2', . . . their values,

$$x = \frac{[al] - [ab]y - [ac]z - \ldots}{[aa]}$$

Similarly if the values of x, z, . . . are supposed to be known, the most probable value of y is found from

$$y = \frac{[bl] - [ba]x - [bc]z - \ldots}{[bb]}$$

and so on for z, . . .

Hence we should obtain the weighted mean values of x, y, . . ., that is, their most probable values, from the simultaneous solution of the equations

$$x = \frac{[al] - [ab]y - [ac]z - \ldots}{[aa]}$$

$$y = \frac{[bl] - [ba]x - [bc]z - \ldots}{[bb]}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

that is, from the simultaneous solution of the equations

$$\begin{aligned} [aa]x + [ab]y + \ldots &= [al] \\ [ba]x + [bb]y + \ldots &= [bl] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \end{aligned} \qquad (5)$$

which equations are equal in number to the number of unknowns. They are called *normal equations*, or, better, *final equations*. We have thus found the most probable values of the unknowns in a series of observation equations by taking the mean.*

* Another method of solving a series of observation equations, due to Richelot, is worthy of notice.

Take the simple case of n equations involving two unknowns. Let the equations ($n > 2$) be written in the strict form

$$\begin{aligned} a_1 x + b_1 y - l_1 &= v_1 \\ a_2 x + b_2 y - l_2 &= v_2 \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ a_n x + b_n y - l_n &= v_n \end{aligned}$$

to find the values of x, y which satisfy them best.

Multiply the equations in order by the undetermined factors k_1, k_2, . . . k_n, and add; then if k_1, k_2, . . . k_n satisfy the condition $[bk] = 0$, we have

$$x = \frac{[kl]}{[ka]} + \frac{[kv]}{[ka]}$$

The best value of x must be that in which the second member is as small as possible, and this

The preceding result may be derived in a manner which will perhaps show still more clearly that we have taken the means of the separate values of the unknowns

For simplicity in writing consider the case of the three observation equations,

$$a_1x + b_1y = l_1$$
$$a_2x + b_2y = l_2$$
$$a_3x + b_3y = l_3$$

already given in Art. 14.

Solve in sets of two in all possible ways by the method of determinants, and

$$(a_1b_2 - a_2b_1)x = b_2l_1 - b_1l_2$$
$$(a_1b_3 - a_3b_1)x = b_3l_1 - b_1l_3$$
$$(a_2b_3 - a_3b_2)x = b_3l_2 - b_2l_3$$

Take the weighted mean of the values of x according to Art. 63, and the result is

$$x = \frac{(b_2l_1 - b_1l_2)(a_1b_2 - a_2b_1) + (b_3l_1 - b_1l_3)(a_1b_3 - a_3b_1) + (b_3l_2 - b_2l_3)(a_2b_3 - a_3b_2)}{(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2}$$

$$= \frac{(a_1l_1 + a_2l_2 + a_3l_3)(b_1b_1 + b_2b_2 + b_3b_3) - (a_1b_1 + a_2b_2 + a_3b_3)(b_1l_1 + b_2l_2 + b_3l_3)}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)(a_1b_1 + a_2b_2 + a_3b_3)}$$

$$= \frac{[al][bb] - [ab][bl]}{[aa][bb] - [ab][ab]}$$

Similarly

$$y = \frac{[al][ab] - [bl][aa]}{[aa][bb] - [ab][ab]}$$

happens when $[ka]$ is as great as possible. But $[ka]$ as a linear function has no maximum unless a condition exists among the k's of at least two dimensions. The simplest such condition is

$$[kk] = 1$$

If we now compute the maximum value of $[ka]$ subject to the conditions

$$[bk] = 0 \quad [kk] = 1$$

we shall find as the best value of x (Todhunter, *Diff. Calc.*, chap. xvi.)

$$x = \frac{[al][bb] - [bl][ab]}{[aa][bb] - [ab][ab]}$$

which agrees with the value resulting from the normal equations 5.

These are the same values as would be derived from the solution of the equations

$$[aa]x+[ab]y=[al]$$
$$[ba]x+[bb]y=[bl]$$

Hence if in a series of n observation equations containing n_i independent unknowns ($n > n_i$) we solve all possible sets of n_i equations by the method of determinants, the weighted means of the separate values found will be the most probable values of the unknowns.

This method of solution would be very troublesome when the number of equations is large, and accordingly we must look for a more convenient but necessarily equivalent method.

Since the principle that the sum of the squares of the residual errors is a minimum holds whether the observed quantity is a function of one or of several unknowns (Art. 17), we can apply it to the simultaneous solution of the equations 2. The residual errors must satisfy the relation

$$v_1^2+v_2^2+\ldots+v_n^2=\text{a min.}$$

that is, we must make

$$(a_1x+b_1y+\ldots-l_1)^2+(a_2x+b_2y+\ldots-l_2)^2 \\ +\ldots+(a_nx+b_ny+\ldots-l_n)^2=\text{a min.}$$

Now, the variables $x, y, \ldots$ being independent, the differential coefficients of the expression for the minimum with respect to each of them in succession must be equal to zero. Hence

$$\begin{aligned} a_1(a_1x+b_1y+\ldots-l_1)+a_2(a_2x+b_2y+\ldots-l_1)+\ldots&=0 \\ b_1(a_1x+b_1y+\ldots-l_1)+b_2(a_2x+b_2y+\ldots-l_2)+\ldots&=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned} \tag{6}$$

or, collecting the coefficients of $x, y, \ldots$ in each equation,

$$\begin{aligned} [aa]x+[ab]y+[ac]z+\ldots&=[al] \\ [ab]x+[bb]y+[bc]z+\ldots&=[bl] \\ [ac]x+[bc]y+[cc]z+\ldots&=[cl] \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned} \tag{7}$$

The number of these equations is the same as the number of unknowns; that is, n_i. Their solution will give the most probable values of $x, y, \ldots$, and, adding these values to the approximate values $X', Y' \ldots$ already known, the most probable values of $X, Y, \ldots$ will result.

The equations 7 are identical in form with equations 5, found by taking the mean of all possible values of x and y. As it has now been abundantly shown that the principle of the mean and that of minimum squares lead to the same results whatever be the number of independent unknowns, we shall use whichever promises to be most convenient in any particular case.

It is useful to notice that equations 6 may be written

$$\begin{aligned}[av] &= 0 \\ [bv] &= 0 \\ \cdot \quad \cdot & \quad \cdot\end{aligned} \tag{8}$$

These relations correspond to $[v] = 0$ in the case of the arithmetic mean, and may be used as a check on the computation of the values of the residuals.

Ex. Given the elevation of Ogden above the ocean by C. P. R. R. levels to be 4301 feet, and the elevation of Cheyenne to be 6075 feet; also the elevation of Cheyenne above Ogden by U. P. R. R. levels to be 1749 feet; find the adjusted elevations of Ogden and Cheyenne above the ocean, supposing the given results to be of equal value.

First solution:

(a) Ogden—

Direct determination	4301	weight	1
Indirect determination	4326	"	½
∴ weighted mean	= 4309 feet.		

(b) Cheyenne—

Direct determination	6075	weight	1
Indirect determination	6050	"	½
∴ weighted mean	= 6067 feet.		

Second solution:

Let X, Y denote the elevations of Ogden and Cheyenne respectively. Then

$$(X - 4301)^2 + (Y - 6075)^2 + (X - Y + 1749)^2 = \text{a min.}$$

Differentiate with respect to X, Y in succession, and

$$2X - Y = 2552$$
$$-X + 2Y = 7824$$
$$\therefore X = 4309 \text{ feet.}$$
$$Y = 6067 \text{ feet.}$$

82. If the observation equations are of different weights $p_1, p_2, \ldots p_n$, then, reducing each equation to the same unit of weight by multiplying it by the square root of its weight, we have

$$\begin{array}{l}\sqrt{p_1}\,a_1x + \sqrt{p_1}\,b_1y + \ldots - \sqrt{p_1}\,l_1 = \sqrt{p_1}\,v_1 \\ \sqrt{p_2}\,a_2x + \sqrt{p_2}\,b_2y + \ldots - \sqrt{p_2}\,l_2 = \sqrt{p_2}\,v_2 \\ \ldots\ldots\ldots\ldots \\ \sqrt{p_n}\,a_nx + \sqrt{p_n}\,b_ny + \ldots - \sqrt{p_n}\,l_n = \sqrt{p_n}\,v_n\end{array} \qquad (1)$$

with

$$[pvv] = \text{a min.}$$

Substituting the values of $\sqrt{p_1}\,v_1$, $\sqrt{p_2}\,v_2$, ... in the minimum equation, and differentiating with respect to $x, y, \ldots$ as independent variables, we have the normal equations

$$\begin{array}{l}[paa]x + [pab]y + \ldots = [pal] \\ [pab]x + [pbb]y + \ldots = [pbl] \\ \ldots\ldots\ldots\end{array} \qquad (2)$$

from which $x, y, \ldots$ may be found.

The relations for weighted equations corresponding to those of Eq. 6, Art. 81, are evidently

$$[pav] = 0,\ [pbv] = 0, \ldots \qquad (3)$$

Formation of the Normal Equations.

83. Instead of forming the minimum equation and differentiating with respect to the unknowns in succession, it is more convenient to proceed according to the following plans suggested by the form of the normal equations themselves.

The first, from equations 6, Art. 81, may be stated as follows: To form the normal equation in x multiply each observation equation by the coefficient of x in that equation, and add the results. To form the normal equation in y multiply each observation equation by the coefficient of y in that equation, and add the results. Similarly for the remaining unknowns.

The second is suggested by the complete form of the normal equations as given in equations 7, Art. 81. According to this plan we compute the quantities $[aa]$, $[ab]$, . . . $[al]$, etc., separately, and write in their proper places in the equations.

The equality of the coefficients of the normal equations in the horizontal and vertical rows leads to a considerable shortening of the numerical work in computing these quantities. Thus with three unknowns, x, y, z, all the unlike coefficients are contained in

$$\begin{aligned} +[aa]x+[ab]y+[ac]z&=[al] \\ +[bb]y+[bc]z&=[bl] \\ +[cc]z&=[cl] \end{aligned}$$

Instead, therefore, of computing 12 quantities, only 9 are necessary, as the remaining 3 can be at once written down. With n unknowns the saving of computation amounts to

$$1+2+3+\ldots+(n-1)=\tfrac{1}{2}n(n-1)$$

quantities.

If the observation equations are of different weights the formation of the normal equations may be carried out precisely in the same way as in the preceding as soon as the observation equations have been reduced to the same unit of weight.

The form of the weighted normal equations, however, shows that it is not necessary, in order to obtain the coefficients $[paa]$, $[pab]$, . . . to multiply the observation equations by the square roots of their weights, and form the aux-

iliary equations 1, Art. 82, since the products aa, ab, . . . multiplied by the weights of the respective equations from which they are formed and summed, give $[paa]$, $[pab]$, . . . directly. This is important because labor-saving.

Ex. 1. Given the observation (or error) equations, all of equal weight,

$$\begin{aligned} x \phantom{{}-y+z} &= 1 \\ x + y \phantom{{}+z} &= 3 \\ x - y + z &= 2 \\ -x - y + z &= 1 \end{aligned}$$

show that the normal equations are

$$\begin{aligned} 4x + y \phantom{{}-2z} &= 5 \\ x + 3y - 2z &= 0 \\ -2y + 2z &= 3 \end{aligned}$$

Ex. 2. The expansions x_1, x_2, x_3, x_4 for 1° Fahr. of four standard of length were found by special experiment to be connected by the following relations at a temperature of 62° Fahr. (μ = one micron.)

$$\begin{array}{llllrlr} +x_1 & & & = & \overset{\mu}{39.945} & \text{weight} & 1 \\ & +x_2 & & = & 5.932 & \text{"} & 16 \\ & & +x_3 & = & 5.371 & \text{"} & 4 \\ & +x_2 - 1.0937x_3 & & = & 0.006 & \text{"} & 8 \\ & +4x_2 & -x_4 & = & -\ 1.335 & \text{"} & 3 \\ +x_1 & & -x_4 & = & +\ 14.833 & \text{"} & 6 \end{array}$$

find their most probable values.

[The normal equations are

$$\begin{aligned} +7x_1 \phantom{{}+72.000x_2 - 8.750x_3} - 6x_4 &= +128.943 \\ +72.000x_2 - 8.750x_3 - 12x_4 &= +\ 78.940 \\ -\ 8.750x_2 + 13.569x_3 \phantom{{}-12x_4} &= +\ 21.432 \\ -6x_1 - 12.000x_2 \phantom{{}+13.569x_3} + 9x_4 &= -\ 84.993 \end{aligned}$$

and $\quad x_1 = \overset{\mu}{39.913}, \quad x_2 = \overset{\mu}{5.932}, \quad x_3 = \overset{\mu}{5.405}, \quad x_4 = \overset{\mu}{25.075}$]

An example will now be given to illustrate the method of forming a series of observation equations :

Ex. 3. At Washington the meridian transits of the following stars were observed to determine the correction and rate of sidereal clock Kessels No. 1324, April 12, 1870, at 11 hours clock time.

Star.	Observed clock time of transit, T.			Right ascension of star, α.
	h.	m.	s.	s.
τ Leonis.	11	21	17.98	16.00
ν Leonis.	11	30	20.41	18.51
β Leonis.	11	42	28.57	26.57
o Virginis.	11	58	38.15	36.20
η Virginis.	12	13	18.37	16.37
θ Virginis.	13	3	16.36	14.39

Let $x =$ corr. of clock at 11 hours clock time,
$y =$ rate per hour of clock.

Now, from theoretical considerations* it is known that the equation

$$x + y(T - 11) = \alpha - T$$

gives the relation between the clock correction and rate and the clock time of transit of each star observed.

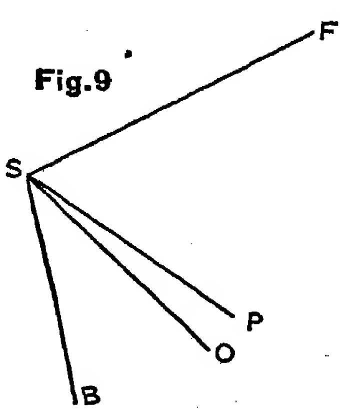
Hence the observation equations are

$$\begin{aligned} x + 0.35y &= -1.98 \\ x + 0.50y &= -1.90 \\ x + 0.71y &= -2.00 \\ x + 0.98y &= -1.95 \\ x + 1.22y &= -2.00 \\ x + 2.05y &= -1.97 \end{aligned}$$

For the remainder of the solution see Art. 104.

Ex. 4. In the triangulation of Lake Superior executed by the U. S. Engineers there were measured at station Sawteeth East the angles

Fig. 9



Farquhar–Porcupine	62° 59′ 40″.33	weight 5
Farquhar–Outer	64° 11′ 34″.92	" 7
Farquhar–Bayfield	100° 20′ 29″.12	" 4
Porcupine–Bayfield	37° 20′ 49″.55	" 7
Outer–Bayfield	36° 08′ 55″.86	" 4

required the adjusted values of the angles.

All of the angles may evidently be expressed in terms of FSP, FSO, FSB. Let X, Y, Z denote the most probable values of these angles, and let X', Y', Z' be assumed approximate values of these most probable values, and x, y, z their most probable corrections. Denoting the measured

* See Chauvenet, *Astronomy*, vol. ii. chap. v.

angles in order by $M_1, M_2, \ldots M_5$, and their most probable corrections by $v_1, v_2, \ldots v_5$, we have

$$\begin{aligned}
X' + x &= \quad X &&= M_1 + v_1\\
Y' + y &= \quad Y &&= M_2 + v_2\\
Z' + z &= \quad Z &&= M_3 + v_3\\
-X' - x + Z' + z &= -X + Z &&= M_4 + v_4\\
-Y' - y + Z' + z &= -Y + Z &&= M_5 + v_5
\end{aligned}$$

For simplicity the assumed approximate values may be taken equal to the observed values of the angles, so that we have the reduced observation equations

$$\begin{array}{lllll}
+x & & & = v_1 & \text{weight } 5\\
& +y & & = v_2 & \quad " \quad 7\\
& & +z & = v_3 & \quad " \quad 4\\
-x & & +z - 0.76 & = v_4 & \quad " \quad 7\\
& -y & +z - 1.66 & = v_5 & \quad " \quad 4
\end{array}$$

Hence the normal equations

$$\begin{aligned}
12x \qquad\quad - 7z &= -\ 5.32\\
+11y - 4z &= -\ 6.64\\
-7x - 4y + 15z &= +11.96
\end{aligned}$$

Solving these equations, we find

$$x = -0''.05, \quad y = -0''.36 \quad z = +0''.68$$

Hence $v_1 = -0''.05$, $v_2 = -0''.36$, $v_3 = +0''.68$, $v_4 = -0''.03$, $v_5 = -0''.62$,

and the adjusted values of the angles are

$$\begin{array}{r}
62^\circ\ 59'\ 40''.28\\
64^\circ\ 11'\ 34''.56\\
100^\circ\ 20'\ 29''.80\\
37^\circ\ 20'\ 49''.52\\
36^\circ\ 08'\ 55''.24
\end{array}$$

We might have used $v_1, v_2, \ldots v_5$ for the corrections without introducing the symbols x, y, z at all.

Ex. 5. If the unknown x occurs in each of the n observation equations

$$\begin{array}{lll}
-x + b_1y + c_1z + \ldots = l_1 & \text{weight} & 1\\
-x + b_2y + c_2z + \ldots = l_2 & \quad " & 1\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \cdot
\end{array}$$

these equations are equivalent to the reduced observation equations

$$\begin{array}{lll}
b_1y + c_1z + \ldots = l_1 & \text{weight} & 1\\
b_2y + c_2z + \ldots = l_2 & \quad " & 1\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \cdot\\
[b]y + [c]z + \ldots = [l] & \quad " & -\dfrac{1}{n}
\end{array}$$

[For the normal equations found from the first set after eliminating x are the same as the normal equations formed from the second set directly.]

Ex. 6. Instead of the observation equation

$$ax + by + cz + \ldots = l \quad \text{weight } p$$

we may write

$$qax + qby + \ldots = ql \quad \text{weight } \frac{p}{q^2}$$

84. **Control of the Formation of the Normal Equations.**—A very convenient check or control is the following. Add as an extra term to each observation equation the sum of the coefficients of x, y, . . . and of the absolute term l in that equation, and treat these added terms just as we do the absolute terms. Thus let s_1, s_2, . . . s_n denote the sums, so that

$$\begin{aligned}
a_1 + b_1 + c_1 + \ldots + l_1 &= s_1 \\
a_2 + b_2 + c_2 + \ldots + l_2 &= s_2 \\
\cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \\
a_n + b_n + c_n + \ldots + l_n &= s_n
\end{aligned}$$

Multiply each of these expressions by its a and add the products, each by its b and add, and so on; then

$$\begin{aligned}
[aa] + [ab] + \ldots + [al] &= [as] \\
[ab] + [bb] + \ldots + [bl] &= [bs] \\
\cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\
[al] + [bl] + \ldots + [ll] &= [ls]
\end{aligned}$$

If these equations are satisfied the normal equations are correct. Thus each normal equation is tested as soon as it is formed.

Since $[aa]$, $[ab]$, . . . $[al]$ have been computed in forming the normal equations, the only new terms to be computed in applying the check are $[as]$, $[bs]$, . . . $[ls]$, $[ll]$.

Various modifications may readily be applied to suit individual tastes. Thus the absolute term may be placed

on the other side of the sign of equality; or the sign of the check may be changed so as to make the sum of each horizontal row equal to zero.

85. **Forms of Computing the Normal Equations.**—When the number of unknowns in the observation equations is large, or when their coefficients contain several figures, it is convenient to have a fixed form for the computation of the terms of the normal equations. It lightens the labor much either in forming, solving, or in finding the precision of the unknowns from these equations, if the computation is so arranged that a check can at all times be applied and the whole process proceed in a uniform and mechanical manner.

The aids in the arithmetical work are a table of squares, a table of products, a table of reciprocals, a table of logarithms, and an arithmometer, or machine for performing multiplications and divisions. The latter is of the greatest use in computations of this kind. With it the *drudgery* of computation is in great measure got rid of. On the Lake Survey two forms of machine were used, the Grant and the Thomas. A series of trials showed that with either machine multiplications could be performed in from one-half to one-third of the time required with a log. table, and with much less liability to error. About as good speed can be made for a short time with Crelle's multiplication-tables, but it cannot be kept up.

Form (a). With Crelle's tables, or with a machine, the products aa, ab, . . . are found directly, and all that is then to be done is to write them in columns and take their sums $[aa]$, $[ab]$, . . . With a Thomas machine, however, each product may be added to all that precede, so that the final values result at once.

Let us, for example, take the observation equations

$$-1.2x+0.2y+0.9=v_1$$
$$+3.0x-2.1y+1.1=v_2$$
$$+0.7x+1.6y-4.0=v_3$$

Arrange as follows, the headings indicating the nature of the numbers underneath :

a	b	l	s
−1.2	+0.2	+0.9	−0.1
+3.0	−2.1	+1.1	+2.0
+0.7	+1.6	−4.0	−1.7

aa	ab	al	as
1.44	−0.24	−1.08	+0.12
9.00	−6.30	+3.30	+6.00
0.49	+1.12	−2.80	−1.19
+10 93	−5.42	−0.58	+4.93
	bb	bl	bs
	0.04	+0.18	−0.02
	4.41	−2.31	−4.20
	2.56	−6.40	−2.72
−5.42	+7.01	−8.53	−6.94
		ll	ls
		0.81	−0.09
		1.21	+2.20
		16.00	+6.80
−0.58	−8.53	+18.02	+8.91

Hence the normal equations, with the check all ready for solution, are

$$o = +10.93x - 5.42y - 0.58 \qquad +4.93$$
$$o = -\ 5.42x + 7.01y - 8.53 \qquad -6.94$$

Form (b). If logarithms alone are used, form a table of the log. coefficients of the observation equations as follows :

$$\log a_1, \log b_1, \ldots \log l_1, \log s_1$$
$$\log a_2, \log b_2, \ldots \log l_2, \log s_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\log a_n, \log b_n, \ldots \log l_n, \log s_n$$

Write $\log a_1$ on a slip of paper and carry it along the top row, forming the products

$$\log a_1a_1, \log a_1b_1, \ldots, \log a_1l_1, \log a_1s_1$$

Similarly with log a_2 form the products from the second row,

$$\log a_2a_2,\ \log a_2b_2,\ .\ .\ .,\ \log a_2l_2,\ \log a_2s_2$$

and so on till log a_n is reached.

The numbers corresponding to these logarithms are next found, so that we have

$$\begin{array}{l} a_1a_1,\ a_1b_1,\ .\ .\ .,\ a_1l_1,\ a_1s_1 \\ a_2a_2,\ a_2b_2,\ .\ .\ .,\ a_2l_2,\ a_2s_2 \\ .\quad .\quad .\quad .\quad . \\ a_na_n,\ a_nb_n,\ .\ .\ .,\ a_nl_n,\ a_ns_n \end{array}$$

By addition we find

$$[aa],\ [ab],\ .\ .\ .\ [al],\ [as]$$

the coefficients of the unknowns in the first normal equation.

Proceed in a precisely similar way with log b_1, log b_2, . . . log b_n, omitting the term $[ab]$ already found; with log c_1, log c_2, . . . log c_n, omitting the terms $[ac]$, $[bc]$ already found; and so on till the last quantity is reached,

log a 0.07918n 0.47712 9.84510	log b 9.30103 0.32222n 0.20412	log l 9.95424 0.04139 0.60206n	log s 9.00000n 0.30103 0.23045n
log aa 0.15836 0.95424 9.69020	log ab 9.38021n 0.79934n 0.04922	log al 0.03342n 0.51851 0.44716n	log as 9.07918 0.77815 0.07555n
	log bb 8.60206 0.64444 0.40824	log bl 9.25527 0.36361n 0.80618n	log bs 8.30103n 0.62325n 0.43457n
		log ll 9.90848 0.08278 1.20412	log ls 8.95424n 0.34242 0.83251

and the numbers corresponding to these logs. are exactly the same as those in form (a). The remainder of the computation is the same as there given.

Form (c). If we wish to use a table of squares altogether, then since

$$ab = \tfrac{1}{2}\{(a+b)^2 - a^2 - b^2\}$$

and therefore

$$[ab] = \tfrac{1}{2}\{[(a+b)^2] - [aa] - [bb]\} \qquad (1)$$

we form the square sums

$$\begin{array}{ccccc} [aa], & [(a+b)^2], & [(a+c)^2], & \ldots & [(a+l)^2], [(a+s)^2] \\ & [bb], & [(b+c)^2], & \ldots & [(b+l)^2], [(b+s)^2] \\ & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ & & & & [ll], [(l+s)^2] \end{array}$$

and perform the necessary subtractions.

In doing this, first take from the table of squares the squares $aa, bb, \ldots ll, ss$, and sum them; next write the coefficients a of x on a slip of paper and carry them over the coefficients of $y, z, \ldots$, forming the sums $a_1 + b_1$, $a_1 + c_1, \ldots$; $a_2 + b_2$, $a_2 + c_2, \ldots$ Take out the squares of these numbers and sum them. Proceed similarly with the coefficients of $y, z, \ldots$ Finish as indicated in (1).

Thus in the preceding example,

aa	bb	ll	ss
1.44	0.04	0.81	0.01
9.00	4.41	1.21	4.00
0.49	2.56	16.00	2.89
10.93	7.01	18.02	6.90

$a+b$	$(a+b)^2$	$a+l$	$(a+l)^2$	$a+s$	$(a+s)^2$
1.0	1.00	0.3	0.09	1.3	1.69
0.9	0.81	4.1	16.81	5.0	25.00
2.3	5.29	3.3	10.89	1.0	1.00
	7.10		27.79		27.69
$[aa] + [bb] =$	17.94	$[aa] + [ll] =$	28.95	$[aa] + [ss] =$	17.83
	−10.84		−1.16		9.86
	− 5.42		−0.58		4.93
	$= [ab]$		$= [al]$		$= [as]$

giving the same results as before.

This form, which is very neat analytically, was first given by Bessel in the *Astron. Nachr.*, No. 399.

A consideration of the simple case of three observation equations, each involving two unknowns, will show that to form the normal equations, using a log. table only, 24 entries in the table are required, while by this method we only need to enter a table of squares 18 times, thus effecting a saving of 6 entries. The Bessel method has also the advantage that, as we deal with squares, all thought with regard to sign is done away with. Besides, if the table of squares is a very extended one, accuracy can be had to a greater number of decimal places than with an ordinary log. table. As compared with the logarithmic form, then, this method is to be preferred, more especially when the coefficients are not very different.

On the other hand, if Crelle's tables or a computing machine is to be had, the direct process explained in (a) is much to be preferred to either, as experience will show.

It is worth noticing that whichever method of formation of the normal equations is adopted, labor will be saved by changing the units in which the unknowns are expressed if the coefficients of the different unknowns are very different. Thus, suppose we had the observation equations,

$$\begin{array}{rr} & \textit{Check sums.} \\ 1000x + 0.0001y = 4.11 & 1004.1101 \\ 999x + 0.0002y = 3.93 & 1002.9302 \\ \ldots\ldots & \ldots \end{array}$$

from which to find x and y.

By placing

$$x' = 100x, \; y' = 0.01y$$

the equations reduce to

$$\begin{array}{rr} & \textit{Check sums.} \\ 10x' \quad + 0.01y' = 4.11 & 14.12 \\ 9.99x' + 0.02y' = 3.93 & 13.94 \\ \ldots\ldots & \ldots \end{array}$$

which are in more manageable shape for solution.

Solution of the Normal Equations.

Before beginning the solution of a series of normal equations we should consider whether the object is to find

(1) the unknowns only, or

(2) the unknowns and their weights;

and, in the latter case,

(a) whether the number of unknowns is large,

(b) whether many of the coefficients of the unknowns in the normal equations are wanting.

Normal equations may be solved by the ordinary algebraic methods for the elimination of linear equations or by the method of determinants. When, however, they are numerous the methods of substitution and of indirect elimination, both introduced by Gauss, are more suitable. Each has its advantages, which will be pointed out as we proceed. The method of substitution is quite mechanical, and is well suited for use with an arithmometer, which is as great a help in solving as it is in forming the normal equations.

86. **The Method of Substitution.**—For convenience in writing, take three unknowns, x, y, z, the process being the same whatever the number.

The normal equations are

$$\begin{aligned}
[aa]x + [ab]y + [ac]z &= [al] \\
[ab]x + [bb]y + [bc]z &= [bl] \\
[ac]x + [bc]y + [cc]z &= [cl]
\end{aligned} \qquad (1)$$

From the first equation

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z + \frac{[al]}{[aa]} \qquad (2)$$

Substitute this value in the remaining equations, and, in the convenient notation of Gauss, there result

$$\begin{aligned}
[bb.1]y + [bc.1]z &= [bl.1] \\
[bc.1]y + [cc.1]z &= [cl.1]
\end{aligned} \qquad (3)$$

where

$$
\left.\begin{aligned}
[bb.1] &= [bb] - \frac{[ab]}{[aa]}[ab] \\
[bc.1] &= [bc] - \frac{[ab]}{[aa]}[ac] \\
[bl.1] &= [bl] - \frac{[ab]}{[aa]}[al] \\
[cc.1] &= [cc] - \frac{[ac]}{[aa]}[ac] \\
[cl.1] &= [cl] - \frac{[ac]}{[aa]}[al]
\end{aligned}\right\} \qquad (4)
$$

Again, from the first of equations 3,

$$
y = -\frac{[bc.1]}{[bb.1]}z + \frac{[bl.1]}{[bb.1]} \qquad (5)
$$

which value substituted in the second equation gives

$$
z = \frac{[cl.2]}{[cc.2]} \qquad (6)
$$

where

$$
\left.\begin{aligned}
[cc.2] &= [cc.1] - \frac{[bc.1]}{[bb.1]}[bc.1] \\
[cl.2] &= [cl.1] - \frac{[bc.1]}{[bb.1]}[bl.1]
\end{aligned}\right\} \qquad (7)
$$

Having thus found z, we have y at once by substituting in (5), and thence x by substituting for y and z their values in (2).

The first equations of the successive groups in the elimination collected are

$$
\begin{aligned}
[aa]x + [ab]y + [ac]z &= [al] \\
[bb.1]y + [bc.1]z &= [bl.1] \\
[cc.2]z &= [cl.2]
\end{aligned} \qquad (8)
$$

These are called the *derived normal equations.*

Divide each of these equations by the coefficient of its first unknown, and

$$
\begin{aligned}
x+\frac{[ab]}{[aa]}y+\frac{[ac]}{[aa]}z&=\frac{[al]}{[aa]}\\
y+\frac{[bc.1]}{[bb.1]}z&=\frac{[bl.1]}{[bb.1]} \qquad (9)\\
z&=\frac{[cl.2]}{[cc.2]}
\end{aligned}
$$

87. **Controls of the Solution.**—In solving a set of normal equations a control is essential. It is sometimes recommended to solve the equations arranged in the reverse order, when, if the work is correct, the same results will be found as before. But what is wanted in a control is a means of checking the work at each step, and not at the conclusion, it may be, of several weeks' work, when, if the results do not agree, all that is known is that there is a mistake somewhere without being able to locate it.

(a) Continuation of the formation control. Experience has shown that it is convenient to carry on through the solution the check used in forming the equations. It merely consists in placing as an extra term to each equation the sums $[as]$, $[bs]$, . . . $[ls]$, and operating on them in the same way as on the absolute terms $[al]$, $[bl]$, . . . The sum of the terms in every line, after each elimination of an unknown, must be each equal to the check sum numerically; the closeness of the agreement depending on the number of decimal places employed.

This check may be applied at every step and mistakes be weeded out.

(b) The diagonal coefficients $[aa]$, $[bb]$, . . . of the normal equations, and $[aa]$, $[bb.1]$, $[cc.2]$, . . . of the derived normal equations, are always positive.

For $[aa]$, $[bb.1]$, . . . being the sums of squares, are positive. Also

$$[aa][bb.1] = \begin{vmatrix} [aa], & [ab] \\ [ab], & [bb] \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1, & b_1 \\ a_3, & b_3 \end{vmatrix}^2 + \dots$$

a positive quantity.

Similarly for $[cc.2]$, $[dd.3]$, . . .

The principle may be of use as a check in the solution of a series of normal equations which are apparently correct, but which have been improperly formed. The following normal equations, which came up once in my own experience, will show this:

$$\begin{aligned} 7x + 7y - 9.1z &= 26 \\ 7x + 28y - 12z &= 64 \\ -9.1x - 12y + 12z &= -39 \end{aligned}$$

The derived normal equations are

$$\begin{aligned} 7x + 7y - 9.1z &= 26 \\ +21y - 2.9z &= 38 \\ -0.2z &= 0.1 \end{aligned}$$

showing by the presence of the negative diagonal term that there was a mistake somewhere. An examination of the data from which the normal equations were derived showed a condition wanting and a condition incorrectly expressed. After the proper corrections had been made the solution was carried through without trouble.

(c) By equations 8, Art. 81, the residuals found by substituting for x, y, z their values in the observation equations must satisfy the relations

$$[av] = [bv] = \dots = 0$$

(d) A very complete check is afforded by the different methods of computing $[vv]$, the sum of the squares of the residuals. (See Art. 100.)

88. **Forms of Solution.**—In applying the method of substitution to any special example it is important that the

arrangement of the computation be convenient and that every step be written down. Experience teaches that simplicity and uniformity of operation are great safeguards against mistakes.

Form (a). Solution without logarithms.

The following form has been found by experience to be convenient. It is well fitted for use with the arithmometer or any other rapid method of multiplication. The form can be readily modified to suit computer's tastes.

For illustration let us take, as before, three unknowns, x, y, z. The computation is divided into sections, each section being formed in a precisely similar way, and in each section one unknown is eliminated.

Given the normal equations,

No.	x	y	z		Check.	Remarks.
I.	$[aa]$	$[ab]$	$[ac]$	$[al]$	$[as]$	
II.	$[ab]$	$[bb]$	$[bc]$	$[bl]$	$[bs]$	
III.	$[ac]$	$[bc]$	$[cc]$	$[cl]$	$[cs]$	

Solution.

IV.	1	$\frac{[ab]}{[aa]}$	$\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$	$\frac{[as]}{[aa]}$	I. ÷ $[aa]$
V.		$[ab]\frac{[ab]}{[aa]}$	$[ab]\frac{[ac]}{[aa]}$	$[ab]\frac{[al]}{[aa]}$	$[ab]\frac{[as]}{[aa]}$	IV. × $[ab]$
II.		$[bb]$	$[bc]$	$[bl]$	$[bs]$	II.
VI.		$[bb.1]$	$[bc.1]$	$[bl.1]$	$[bs.1]$	II. − V.
VII.		$[ac]\frac{[ab]}{[aa]}$	$[ac]\frac{[ac]}{[aa]}$	$[ac]\frac{[al]}{[aa]}$	$[ac]\frac{[as]}{[aa]}$	IV. × $[ac]$
III.		$[bc]$	$[cc]$	$[cl]$	$[cs]$	III.
VIII.		$[bc.1]$	$[cc.1]$	$[cl.1]$	$[cs.1]$	III. − VII.
IX.		1	$\frac{[bc.1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$	$\frac{[bs.1]}{[bb.1]}$	VI. ÷ $[bb.1]$
X.			$[bc.1]\frac{[bc.1]}{[bb.1]}$	$[bc.1]\frac{[bl.1]}{[bb.1]}$	$[bc.1]\frac{[bs.1]}{[bb.1]}$	IX. × $[bc.1]$
VIII.			$[cc.1]$	$[cl.1]$	$[cs.1]$	
XI.			$[cc.2]$	$[cl.2]$	$[cs.2]$	VIII. − X.
			1	$\frac{[cl.2]}{[cc.2]}$	$\frac{[cs.2]}{[cc.2]}$	XI. ÷ $[cc.2]$

$$\therefore z = \frac{[cl.2]}{[cc.2]}$$

From Eq. IX.

$$y = -z\frac{[bc.1]}{[bb.1]} + \frac{[bl.1]}{[bb.1]}$$

From Eq. IV.

$$x = -y\frac{[ab]}{[aa]} - z\frac{[ac]}{[aa]} + \frac{[ab]}{[aa]}$$

To eliminate the first unknown, x. In the first line write the quotients $\frac{[ab]}{[aa]}, \frac{[ac]}{[aa]}, \ldots$ that is, the coefficients of the first normal equation divided by $[aa]$, the coefficient of x in that equation.

The first line is now multiplied in order by $[ab]$, $[ac]$, forming the second and fifth lines.

In the third and sixth lines write equations II. and III.

The fourth line is the sum of the second and third, and the seventh the sum of the fifth and sixth.

This concludes the elimination of x, and the results in the fourth and seventh lines involve y and z only.

Take now these results and proceed in a precisely similar way to eliminate y.

The value of the last unknown, z, next results.

Now proceed to find y and x. Thus substitute for z its value in the eighth line, and we have y; and for y and z their values in the first line, and we have x.

In carrying this solution into practice there are three points that deserve special notice:

(1) In order to render the work mechanical, and so lighten the labor, the number of different operations should be made as small as possible. Instead, therefore, of dividing by $[aa]$, $[bb.1]$, $[cc.2]$, it is better to multiply by the reciprocals of these quantities, and, in order to avoid subtractions, to first change the signs of the reciprocals. We shall then have to perform only two simple operations—multiplication and addition. By transferring the terms $[al]$, $[bl]$, $[cl]$ to the left-hand side of the equations before beginning the solution, the values of the unknowns will come out with their proper signs.

(2) Equations VI. and VIII. are the normal equations with x eliminated. An inspection of them shows that the coefficients of the unknowns follow the same law as the coefficients of the unknowns in the original normal equations with respect to symmetry of vertical and horizontal columns. Hence in the elimination it is unnecessary to compute these common terms more than once. Thus $[bc.1]$ from Eq. VI. may be written down as the first term of Eq. VIII. This principle is of great use in shortening the work when the number of unknowns is large.

(3) In a numerical example it is evident that since $[aa]$, $[bb.1]$, $[cc.2]$ do not in general divide exactly into the other coefficients of their respective equations, and that only approximate values of the unknowns can at best be obtained, it will give a closer result to divide by the larger coefficients and multiply by the smaller than *vice versa*. Attention to this by a proper arrangement of the coefficients before beginning the solution results in a considerable saving of labor, as the successive coefficients in the course of the elimination need not be carried to as many places of decimals to insure the same accuracy that a different arrangement would require.

Ex. To make the preceding perfectly plain we shall solve in full the normal equations formed in Art. 85.

(1) Write the absolute term on the right of the sign of equality, and make the check sum equal to the sum of the other terms in each horizontal row.

	x	y	z	Check.	Remarks.
I. II.	+ 10.93 − 5.42	− 5.42 + 7.01	+ 0.58 + 8.53	+ 6.09 + 10.12	
III.	+ 1.	− 0.496	+ 0.053	+ 0.557	I. ÷ 10.93
IV. II.		+ 2.688 + 7.01	− 0.288 + 8.53	− 3.019 + 10.120	III. × − 5.42 II.
V.		+ 4.322	+ 8.818	+ 13.139	II. − IV.
VI.		+ 1.	+ 2.040 = y	+ 3.040	V. ÷ 4.322
VII. III.	+ 1.	− 0.496 − 0.496	− 1.012 + 0.053	− 1.508 + 0.557	VI. × − 0.496 III.
VIII.	+ 1.	0.	+ 1.065 = x	+ 2.065	III. − VII.

Hence

$$x = 1.06 \qquad y = 2.04$$

(2) Write the constant term on the left of the sign of equality, and form the check so as to make the sum of the terms in each horizontal line equal to zero.

	Reciprocals.	x	y	l	Check.	Remarks.
I. II.	0.0915	+ 10.93 − 5.42	− 5.42 + 7.01	− 0.58 − 8.53	− 4.93 + 6.94	
III.		− 1.	+ 0.496	+ 0.053	+ 0.451	I. × − 0.0915
IV. II.			− 2.688 + 7.010	− 0.288 − 8.530	− 2.445 + 6.940	III. × − 5.42 II.
V.	0.2314		+ 4.322	− 8.818	+ 4.495	II. + IV.
VI.			− 1.	+ 2.040 $= y$	− 1.040	V. × − 02314.
VII. III.		− 1.	− 0.496 + 0.496	+ 1.012 + 0.053	− 0.516 + 0 451	VI. × 0.496 III.
VIII.		− 1.		+ 1.065 $= x$	− 0.065	III. + VII.

In order to find the values of the unknowns to two places of decimals the computation should be carried through to three places, and the third place dropped in the final result.

Form (b). The logarithmic solution.

As an example of the logarithmic method let us take the general form of the preceding example, when R and S are substituted for the absolute terms 0.58 and 8.53 respectively.

In the numerical work it is better to convert all the divisions into multiplications. Therefore write down the complementary logs. of the divisors with the signs changed. Each multiplier may now be written as needed on a slip of paper and carried over each logarithm to be operated on. Thus for the first operation the slip would have on it 8.96138n, where the n indicates that the number is negative.

Paper ruled into small squares, so as to bring the figures in the same vertical columns and facilitate additions and subtractions, renders the work more mechanical and is consequently an assistance to the computer.

In general solutions, when the number of unknowns is large, it will be found much better to carry a double check, one for the coefficients of x, y, . . . and the other for the coefficients of R, S, . . . Though unnecessary in our example, it is inserted for illustration.

It will be noticed that the coefficients of R, S in the values of x, y follow the same law of symmetry as the normal equations. A little consideration will show that this is always the case.

Hence, attending to this, we may shorten the computation by leaving out the common terms. We have, therefore, one term less to compute for each unknown, proceeding from the last to the first. The case is precisely analogous to that of Art. 83.

	x	y	Check.	R	S	Check.	Remarks.
I.	10 93	− 5.42	− 5.51	− 1.		+ 1.	
II.	− 5.42	+ 7.01	− 1.59		− 1.	+ 1.	
III.	1.03862	0.73400*n*	0.74115*n*	0.		0.	log I.
IV.	(8.96138*n*)	9.69538	9.70253	8.96138		8.96138*n*	III. − log 10.93
				+ 0.091		− 0.091	Nos.
V.		0.42938*n*	0.43653*n*	9.69538*n*		9.69538	IV. − log 5.42
VI.		− 2.668	− 2.732	− 0.496		+ 0.496	Nos.
II.		+ 7.010	− 1.590		− 1.	+ 1.	II.
VII.		+ 4.322	− 4.322	− 0.496	− 1.	+ 1.496	VI. + II.
VIII.		0.63568	0.63568*n*	0.69548*n*	0.0.	0.17493	log VII.
IX.		(9.36432*n*)	0.00000	9.05980	9.36432	9.53925*n*	VIII. − log 4.322
X.		− 1.	+ 1.	+ 0.115	+ 0 231	− 0.346	Nos.
XI.				8.75518	9.05970	9.23463*n*	IX. + log $\frac{5.42}{10.93}$
XII.				+ 0.057	+ 0.115	− 0.172	Nos.
				+ 0.091		− 0.091	
XIII.	− 1.		+ 1.	+ 0.148	+ 0.115	− 0.263	

$$\therefore\ x = 0.148R + 0.115S$$
$$y = 0.115R + 0.231S$$

Substituting for R and S their values, we have, as before,

$$x = 1.06$$
$$y = 2.04$$

Ex. 1. In the elimination of n normal equations by the method of substitution, show that the total number of independent coefficients in the original and derived normal equations is $\dfrac{n(n+1)(n+5)}{6}$.

[The sum is $\frac{1}{2}\{1\,.\,4 + 2\,.\,5 + \ldots n(n+3)\}$]

Ex. 2. If the elimination of the unknowns in the normal equations is carried out by the method of substitution, the product

$$[aa]\,[bb.1]\,[cc.2]\ \ldots$$

has the same value whatever order has been followed.

[For it is the determinant $\begin{vmatrix} [aa], & [ab], & [ac] & \ldots \\ [ab], & [bb], & [bc] & \ldots \\ [ac], & [bc], & [cc] & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$

89. **The Method of Indirect Elimination.**—This method is often of service when the number of unknowns is large and many of the terms in the normal equations are wanting. The principle involved in the solution is very simple. If x', y', z' are approximate values of x, y, z, and x_1, y_1, z_1 the corrections to these values, so that

$$x = x' + x_1$$
$$y = y' + y_1$$
$$z = z' + z_1$$

then, substituting these values in the normal equations, we have

$$[aa]x_1 + [ab]y_1 + [ac]z_1 = [al]_1$$
$$[ab]x_1 + [bb]y_1 + [bc]z_1 = [bl]_1$$
$$[ac]x_1 + [bc]y_1 + [cc]z_1 = [cl]_1$$

where the new absolute terms $[al]_1$, $[bl]_1$, $[cl]_1$ will be, on the whole, smaller than the original terms $[al]$, $[bl]$, $[cl]$. A second approximation will tend to decrease the absolute terms still farther. The approximations are continued till the absolute terms either vanish or are sufficiently small. Then

$$x = x' + x'' + \ldots; \; y = y' + y'' + \ldots; \ldots$$

In finding the approximate values it is best to consider one unknown at a time, and preferably the equations should be operated upon in the order of magnitude of the absolute terms. Thus suppose $[al]$ the largest absolute term; then, since the diagonal term is in general the important term in a normal equation, we may neglect all but it in the equation and call $\frac{[al]}{[aa]}$ the first approximation to the value of x. Substitute for x this value in the three equations, when a

value of y or of z may be similarly found. Or the form of the equations may be such that first approximations to the values of both x and y, or of x, y, and z, may be advantageously found from the original equations before making any substitutions. These and similar points must be decided according to the circumstances of the case in hand.

If the diagonal terms in the normal equations, instead of being much larger than the other terms, are but little larger than several of them, considerable difficulty will be found in making the approximations. An approximation made for one unknown may counterbalance those made for several others, and the whole process will be found tedious and troublesome. Various expedients have been suggested for getting over this difficulty; but in all cases where the normal equations are not very loosely connected (that is, where many terms are wanting) and the diagonal coefficients large, my experience has been that it is much better to use the method of substitution, or, in simple cases, the ordinary algebraic methods of elimination.

While employed on the adjustment of the primary triangulation of Lake Superior we came across Von Freeden's glowing account* of the advantages of this method of solution, and determined to give it a trial. The number of equations to be solved was 32. Where the connection of the unknowns was loose it worked well enough, but where they fell in groups it was very troublesome and slow. On the whole, the work was more fatiguing and took longer time. We never tried the method again, though, as it works so nicely in simple cases, we were at one time very much in its favor.

Gauss, the author of this form of solution, describes it as in general tedious (*langwierig*), and very justly. For the contrary view see *Coast Survey Report*, 1855, App. 44; Von

* *Die Praxis der Methode der kleinsten Quadrate.* Braunschweig, 1863.

Freeden, loc. cit.; Vogler, *Grundzüge der Ausgleichungsrechnung*, p. 129.

Ex. Required to solve the equations formed in Art. 85.

Solution.

		Check.	x	y
10.93	− 5.42	− 5.51		
− 5.42	+ 7.01	− 1.59		
− 0.58	− 8.53	+ 9.11		
− 5.420	+ 7.01	− 1.59		1.0
+ 5.465	− 2.71	− 2.755	0.5	
− 0.535	− 4.230	+ 4.765		
+ 5.465	− 2.71	− 2.755		1.0
− 5.420	+ 7.01	− 1.59	0.5	
− 0.490	+ 0.070	+ 0.420		
+ 0.5465	− 0.271	− 0.2755	0.05	
− 0.1626	+ 0.2103	− 0.0477		0.03
− 0.1061	+ 0.0093	+ 0.0968		
+ 0.1093	− 0.0542	− 0.0551	0.01	
− 0.0542	+ 0.0701	− 0.0159		0.01
− 0.0510	+ 0.0252	+ 0.0258		
+ 0.0546	− 0.0271	− 0.0275	0.005	
+ 0.0036	− 0.0019	− 0.0017	1.065	2.04

The first two lines contain the coefficients of x, y, together with the check sums; the third line the absolute terms and their check sum.

The first approximations are taken $y = 1$ and $x = 0.5$. The second and first lines are multiplied by these numbers and the products written in the fourth and fifth lines. The third, fourth, and fifth lines are summed, and $y = 1$, $x = 0.5$ are taken as the next approximations. The sum of the x column of approximations gives the value of x, and of the y column the value of y.

90. **Combination of the Direct and Indirect Methods of Solution.**—A numerical example illustrating an ingenious combination of the direct and indirect methods of elimination is given in the *Coast Survey Report* for 1878,

Appendix 8. For performing the multiplications necessary Crelle's tables are used altogether.

In order to make the process employed readily followed I will give in general terms the solution of the three normal equations

$$[aa]x + [ab]y + [ac]z = [al]$$
$$[ab]x + [bb]y + [bc]z = [bl]$$
$$[ac]x + [bc]y + [cc]z = [cl]$$

according to this form.

The coefficients and absolute term of the first equation are written in line 1, Table A. The reciprocal of the diagonal coefficient $[aa]$ is taken from a table of reciprocals and entered in the front column with the minus sign prefixed. The remaining terms of line 1 are multiplied by this reciprocal, and the products written in line 2. This gives x as an explicit function of y and z.

The coefficients and absolute term of Eq. 2 (omitting the coefficient of x) are written in line 1, Table B. The terms in line 2, Table A, beginning with that under y, are multiplied by $[ab]$, the coefficient of y, and the products set down in line 2, Table B. The sum of lines 1, 2, Table B, is now written in line 3, Table A.

Line 4, Table A, is found from line 3 in the same way as line 2 was found from line 1. This gives y as an explicit function of z.

The coefficients and absolute term of Eq. 3 (omitting the coefficients of x and y) are written in line 3, Table B. The terms in lines 2, 4, Table A, beginning with those under z, are multiplied by $[ac]$, $[bc.1]$, the coefficients of z in lines 1, 3 respectively, and the products set down in lines 4, 5, Table B. The sum of lines 3, 4, 5, Table B, is written in line 5, Table A.

Line 6, Table A, gives the value of z.

The next step is to find y and x. The coefficients of the explicit functions are written in Table C. The absolute terms of the explicit functions are written in the first

line of Table D. The value of z is multiplied by the coefficients of z in Table C, and the products written in the second line of Table D. The sum of the numbers in column y gives the value of y written underneath in line 3. The value of y is multiplied by the coefficients of y in Table C, and the products written in the third line of Table D. The sum of the numbers in column x gives the value of x.

The values of x, y, z are now found to three places of decimals. Denote them by x', y', z'. If these values are not sufficiently close a second approximation must be made. This we proceed to describe.

First substitute the values obtained in the original normal equations, and carry out to a sufficient number of decimal places. The residuals are written in the first line of Table E. The coefficients in line 1, Table C, are multiplied by $-[al]_1$, and the products written in line 2, Table E. The first reciprocal in Table A is multiplied by the same residual, and the product written in column x, line 1, Table F. The sum of the numbers in column 2, Table E, is written underneath, as $-[bl.1]_1$.

The coefficient in line 2, Table C, is multiplied by $-[bl.1]_1$ and the product written in line 3, Table E. The second reciprocal in Table A is multiplied by the same residual, and the product written in column y, line 1, Table F. The sum of the numbers in column 3, Table E, is written underneath, as $-[cl.2]_1$.

The third reciprocal of Table A is multiplied by this residual, and the product written in column z, line 1, Table F. This gives the correction to the value of z. The first line of Table F corresponds to the first line of Table D, and exactly the same process employed in Table D will complete Table F. The total values now are

$$x = x' + x''$$
$$y = y' + y''$$
$$z = z' + z''$$

A

Recip.	x	y	z	
	$+[aa]$	$+[ab]$	$+[ac]$	$-[al]$
$-\frac{1}{[aa]}$	$x =$	$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$	$+\frac{[al]}{[aa]}$
		$[bb.1]$	$+[bc.1]$	$-[bl.1]$
$-\frac{1}{[bb.1]}$		$y =$	$-\frac{[bc.1]}{[bb.1]}$	$+\frac{[bl.1]}{[bb.1]}$
			$+[cc.2]$	$-[cl.2]$
$-\frac{1}{[cc.2]}$			$z =$	$+\frac{[cl.2]}{[cc.2]}$

B

y	z	
$+[bb]$	$+[bc]$	$-[bl]$
$-\frac{[ab]}{[aa]}[ab]$	$-\frac{[ac]}{[aa]}[ab]$	$+\frac{[ab]}{[aa]}[ab]$
	$[cc]$	$-[cl]$
	$-\frac{[ac]}{[aa]}[ac]$	$+\frac{[al]}{[aa]}[ac]$
	$-\frac{[bc.1]}{[bb.1]}[bc.1]$	$+\frac{[bl.1]}{[bb.1]}[bc.1]$

C

y	z
$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$
	$-\frac{[bc.1]}{[bb.1]}$

D

x	y	z
$+\frac{[al]}{[aa]}$	$+\frac{[bl.1]}{[bb.1]}$	$+\frac{[cl.2]}{[cc.2]}$
$-\frac{[ac]}{[aa]}z'$	$-\frac{[bc.1]}{[bb.1]}z'$	z'
$-\frac{[ab]}{[aa]}y'$	y'	
x'		

E

x	y	z
$-[al]_1$	$-[bl]_1$	$-[cl]_1$
	$+\frac{[ab]}{[aa]}[al]_1$	$+\frac{[ac]}{[aa]}[al]_1$
	$-[bl.1]_1$	$+\frac{[bc.1]}{[bb.1]}[bl.1]_1$
		$-[cl.2]_1$

F

x	y	z
$+\frac{[al]_1}{[aa]}$	$+\frac{[bl.1]_1}{[bb.1]}$	$+\frac{[cl.2]_1}{[cc.2]}$
$-\frac{[ac]}{[aa]}z''$	$-\frac{[bc.1]}{[bb.1]}z''$	z''
$-\frac{[ab]}{[aa]}y''$	y''	
x''		

Addition of New Equations.—It often happens that after the adjustment of a long series of observations additional observations are made leading to additional condition equations. To make a solution *de novo* is necessary, but the work may be very materially shortened by the process just given. Suppose, for simplicity, that one new condition has been established. This will give one additional normal equation, which may be written

$$[ad]x + [bd]y + [cd]z + [dd]w = [dl] \qquad (1)$$

w being the new unknown.

The extra term to each of the other normal equations may be written down at sight. The complete equations are

$$\begin{array}{llll}
[aa]x + [ab]y + [ac]z + [ad]w = [al] \\
 + [bb]y + [bc]z + [bd]w = [bl] \\
 + [cc]z + [cd]w = [cl] \\
 + [dd]w = [dl]
\end{array}$$

Now, values of x, y, z have been already found from the normal equations resulting from the original condition equations, and these values may be taken as first approximations to the values of x, y, z resulting from the above four normal equations. Substitute in (1), and

$$[ad]x' + [bd]y' + [cd]z' + [dd]w = [dl]' \qquad (2)$$

where x', y', z' are corrections to the approximate values of x, y, z. The solution is now finished as follows:

Form Table C (a) by adding the extra column w to Table C. The term $-\frac{[ad]}{[aa]}$ is found by multiplying $[ad]$ by the first reciprocal. The coefficients of the new equation, (2), are written in the first line of Table G. Since corrections to values already found are required, the method of proceeding must be similar to that employed in Table E. The notation in Tables C (a) and G explains this.

The reciprocal of the sum of column w, that is, $[dd.3]$, in

Table G is written last in the column of reciprocals of Table C (a) with the minus sign. The product of this reciprocal and the absolute term $-[dl]'$ of the new equation, that is, $\frac{[dl]'}{[dd.3]}$, is an approximate value of w. This value of w is multiplied by the terms in the last column of Table C (a), and the products are written in the first line of Table H. Column z gives the correction to z. Table H is now completed in the same way as Tables D and F.

C (a)

	y	z	w
$-\frac{1}{[aa]}$	$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$	$-\frac{[ad]}{[aa]}$
$-\frac{1}{[bb.1]}$		$-\frac{[bc.1]}{[bb.1]}$	$-\frac{[bd.1]}{[bb.1]}$
$-\frac{1}{[cc.2]}$			$-\frac{[cd.2]}{[cc.2]}$
$-\frac{1}{[dd.3]}$			

G

x	y	z	w
$[ad]$	$[bd]$	$[cd]$	$[dd]$
	$-\frac{[ab]}{[aa]}[ad]$	$-\frac{[ac]}{[aa]}[ad]$	$-\frac{[ad]}{[aa]}[ad]$
	$[bd.1]$	$-\frac{[bc.1]}{[bb.1]}[bd.1]$	$-\frac{[bd.1]}{[bb.1]}[bd.1]$
		$[cd.2]$	$-\frac{[cd.2]}{[cc.2]}[cd.2]$
			$[dd.3]$

H

x	y	z	w
$-\frac{[ad]}{[aa]}w$	$-\frac{[bd.1]}{[bb.1]}w$	$-\frac{[cd.2]}{[cc.2]}w$	$+\frac{[dl]'}{[dd.3]}$
$-\frac{[ac]}{[aa]}z'$	$-\frac{[bc.1]}{[bb.1]}z'$	z'	w
$-\frac{[ab]}{[aa]}y'$	y'		
x'			

91. Time required to Solve a Set of Equations.— The labor involved in solving a series of normal equations, and the consequent time employed, increases enormously with an increase in the number of normal equations. To any one who has never been engaged in such work it will

seem out of all reason. Thus Dr. Hügel,* of Hessen, Germany, states that he has solved 10 normal equations in from 10 to 12 hours, using a log. table, but that 29 equations took him 7 weeks.

The following are examples of rapid work: Gen. Baeyer, in the *Küstenvermessung* (Vorwort, p. vii.) mentions that Herr Dase solved 86 normal equations between the first of June and the middle of September; and Mr. Doolittle,† of the U. S. Coast Survey, solved 41 normal equations in 5½ days, or 36 working hours.

A great deal depends, so far as speed is concerned, on the form of solution and on the mechanical aids used. With a machine or with Crelle's tables much better time can be made than by the logarithmic method, which is by far the most roundabout. Mr. Doolittle used Crelle's tables and the form explained in the preceding article.

In comparisons such as these it is important to know how many of the terms of the normal equations are wanting. Herr Dase's and Mr. Doolittle's equations were both derived from triangulation work. The latter had only 430 terms in his 41 equations, while the former had 3141 different terms in his 86 equations.

A machine for the solution of simultaneous linear equations has been invented by Sir W. Thomson, of Glasgow, Scotland. The results are obtained, by a series of approximations, to any accuracy required. A description of the machine and of the mathematical principles underlying its construction will be found in Thomson and Tait's *Natural Philosophy*, vol. i. Appendix B. Sir W. Thomson says: "The exceeding ease of each application of the machine promises well for its real usefulness, whether for cases in which a single application suffices or for others in which the requisite accuracy is reached after two or three or more of successive approximations."

With regard to a machine for performing multiplica-

* *General-Bericht über die europäischen Gradmessung*, 1867, p. 109.

† *Report U. S. Coast Survey*, 1878, Appendix 8.

tions and divisions, I am certain that no one will ever prefer to use a log. table in solving a set of equations by the method of substitution who has ever used any of the forms of arithmometer.

92. For Jacobi's method of elimination, and the application of determinants to the solution of normal equations, see *Astron. Nachr.*, 404, 1960; *Month. Not. Roy. Astron. Soc.*, vols. xxxiv., xl.; *Proc. Amer. Assoc. for Adv. of Science*, 1881.

The Precision of the Most Probable (Adjusted) Values.

93. The problem now before us is to find the m. s. e. of the unknowns $x, y, \ldots$ as determined from a series of normal equations. If the observation equations are reduced to the same unit of weight, which we shall take to be unity for convenience, the general form of the normal equations is

$$\begin{array}{l} [aa]x + [ab]y + \ldots = [al] \\ [ab]x + [bb]y + \ldots = [bl] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (1)$$

Let $\mu =$ the m. s. e. of a single observation.

$\mu_x, \mu_y, \ldots =$ the m. s. e. of $x, y, \ldots$

$p_x, p_y, \ldots =$ the weights of $x, y, \ldots$

From Art. 56 we have

$$p_x\mu_x^2 = p_y\mu_y^2 = \ldots = \mu^2 \qquad (2)$$

In order, therefore, to determine $\mu_x, \mu_y, \ldots$ we must make two computations, one of the weights $p_x, p_y, \ldots$ and the other of μ, the m. s. e. of a single observation.

It is evident from an inspection of the normal equations that $x, y, \ldots$ are linear functions of $l_1, l_2, \ldots$ Let, then,

$$\begin{array}{l} x = \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_n l_n = [\alpha l] \\ y = \beta_1 l_1 + \beta_2 l_2 + \ldots + \beta_n l_n = [\beta l] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (3)$$

in which $\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots; \ldots$ are functions of $a_1, b_1, \ldots; a_2, b_2, \ldots; \ldots$ their values being as yet undetermined.

Now, μ being the m. s. e. of each of the observed quantities $M_1, M_2, \ldots M_n$, must be also the m. s. e. of $l_1, l_2, \ldots l_n$, which differ from $M_1, M_2, \ldots M_n$ by known amounts (see Art. 81). Hence since $l_1, l_2, \ldots l_n$ are independent of each other (Art. 19),

$$\mu_x^2 = \mu^2 [\alpha\alpha], \ \mu_y^2 = \mu^2 [\beta_1\beta_1], \ \ldots \tag{4}$$

and therefore

$$p_x = \frac{1}{[\alpha\alpha]}, \ p_y = \frac{1}{[\beta_1\beta_1]}, \ \ldots \tag{5}$$

We shall first of all determine the weights $p_x, p_y, \ldots$

Before proceeding with the general proof let us consider the simple case of two unknowns, x, y, which are to be found from the three observation equations

$$\begin{aligned} a_1x + b_1y &= l_1 \\ a_2x + b_2y &= l_2 \\ a_3x + b_3y &= l_3 \end{aligned}$$

all of the same weight.

The normal equations are

$$\begin{aligned} [aa]x + [ab]y &= [al] \\ [ab]x + [bb]y &= [bl] \end{aligned}$$

Their solution by the method of substitution gives

$$\begin{aligned} y &= \frac{[bl.1]}{[bb.1]} \\ &= \frac{1}{[bb.1]}\left([bl] - \frac{[ab]}{[aa]}[al]\right) \\ &= \frac{1}{[bb.1]}\left(b_1 - \frac{[ab]}{[aa]}a_1\right)l_1 + \left(b_2 - \frac{[ab]}{[aa]}a_2\right)l_2 + \left(b_3 - \frac{[ab]}{[aa]}a_3\right)l_3 \end{aligned}$$

Hence since l_1, l_2, l_3 are independent of one another,

$$\begin{aligned} \frac{1}{p_y} &= \frac{1}{[bb.1]^2}\left(b_1 - \frac{[ab]}{[aa]}a_1\right)^2 + \left(b_2 - \frac{[ab]}{[aa]}a_2\right)^2 + \left(b_3 - \frac{[ab]}{[aa]}a_3\right)^2 \\ &= \frac{1}{[bb.1]} \end{aligned} \tag{6}$$

that is, *the weight of y is the denominator of the value of y found from the solution of the normal equations by the Gaussian method of substitution.*

The general demonstration may be carried out more simply by the application of the principles of undetermined coefficients. Thus substitute $[\alpha l]$, $[\beta l]$, . . . for x, y, . . . in the normal equations (1), and

$$\begin{aligned} [aa][\alpha l] + [ab][\beta l] + \ldots &= [al] \\ [ab][\alpha l] + [bb][\beta l] + \ldots &= [bl] \\ \ldots \quad \ldots \quad \ldots \end{aligned} \tag{7}$$

or, arranging according to l_1, l_2, . . .

$$\{[aa]\alpha_1 + [ab]\beta_1 + \ldots - a_1\}l_1 + \{[aa]\alpha_2 + [ab]\beta_2 + \ldots - a_2\}l_2 + \ldots = 0$$

$$\{[ab]\alpha_1 + [bb]\beta_1 + \ldots - b_1\}l_1 + \{[ab]\alpha_2 + [bb]\beta_2 + \ldots - b_2\}l_2 + \ldots = 0$$

$$\ldots \quad \ldots \quad \ldots$$

The unknown quantities α_1, α_2, . . . may be so determined that the coefficients of l_1, l_2, . . . shall each equal zero. Hence the several sets of equations

$$\left\{\begin{aligned} [aa]\alpha_1 + [ab]\beta_1 + \ldots - a_1 &= 0 \\ [aa]\alpha_2 + [ab]\beta_2 + \ldots - a_2 &= 0 \\ \ldots \quad \ldots \quad \ldots \end{aligned}\right. \tag{8}$$

$$\left\{\begin{aligned} [ab]\alpha_1 + [bb]\beta_1 + \ldots - b_1 &= 0 \\ [ab]\alpha_2 + [bb]\beta_2 + \ldots - b_2 &= 0 \\ \ldots \quad \ldots \quad \ldots \end{aligned}\right.$$

$$\ldots \quad \ldots \quad \ldots$$

are simultaneously satisfied by the same values of

$$\alpha_1\ \alpha_2, \ldots;\ \beta_1,\ \beta_2, \ldots;\ \ldots$$

Multiply the equations of each set by $a_1, a_2, \ldots$ in order, and add; then necessarily

$$[a\alpha] = 1, \; [a\beta] = 0, \; [a\gamma] = 0, \ldots \tag{9}$$

In a similar way, multiplying by $b_1, b_2, \ldots$; $c_1, c_2, \ldots$, etc., and adding, there result

$$\begin{array}{l} [b\alpha] = 0, \; [b\beta] = 1, \; [b\gamma] = 0, \ldots \\ [c\alpha] = 0, \; [c\beta] = 0, \; [c\gamma] = 1, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Again, multiply the first set by $\alpha_1, \alpha_2, \ldots$, the second by $\beta_1, \beta_2, \ldots$, and so on, and add, and we have the sets of equations

$$\begin{array}{l} \left\{ \begin{array}{l} [aa][a\alpha] + [ab][a\beta] + \ldots = [a\alpha] = 1 \\ [ab][a\alpha] + [bb][a\beta] + \ldots = [b\alpha] = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right. \\ \left\{ \begin{array}{l} [aa][\alpha\beta] + [ab][\beta\beta] + \ldots = [a\beta] = 0 \\ [ab][\alpha\beta] + [bb][\beta\beta] + \ldots = [b\beta] = 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \tag{10}$$

from which equations $[\alpha\alpha]$, $[\alpha\beta]$, . . . may be found.

It is plain that the coefficients of $[\alpha\alpha]$, $[\alpha\beta]$, . . . ; $[\alpha\beta]$, $[\beta\beta]$, . . . in these equations are the same as those of x, y, . . . in the normal equations, and that the absolute terms are 1, 0, . . . ; 0, 1, . . . ; . . . instead of $[al]$, $[bl]$, . . . Hence,

94. **To Find the Weights of the Unknowns.**—*In the first normal equation write* 1 *for* $[al]$, *and in the other normal equations put* 0 *for each of* $[bl]$, $[cl]$, . . . ; *the value of* x *found from these equations will be the reciprocal of the weight of* x, *and the values of* y, z, . . . *will be the values of* $[\alpha\beta]$, $[\alpha\gamma]$, . . . *In the second normal equation write* 1 *for* $[bl]$, *and in the other equations put* 0 *for each of* $[al]$, $[cl]$, . . . ; *the value of* y *found from these equations will be the reciprocal of the weight of* y, *and the values of* z, . . . *found will be the*

values of $[\beta\gamma]$, . . . Similarly for each of the unknowns in succession. For example, the weight equations for three unknowns are

$[\alpha\alpha]$	$[\alpha\beta]$	$[\alpha\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	1
$+[ab]$	$+[bb]$	$+[bc]$	0
$+[ac]$	$+[bc]$	$+[cc]$	0

$[\alpha\beta]$	$[\beta\beta]$	$[\beta\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	0
$+[ab]$	$+[bb]$	$+[bc]$	1
$+[ac]$	$+[bc]$	$+[cc]$	0

$[\alpha\gamma]$	$[\beta\gamma]$	$[\gamma\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	0
$+[ab]$	$+[bb]$	$+[bc]$	0
$+[ac]$	$+[bc]$	$+[cc]$	1

The quantities $[\alpha\beta]$, $[\alpha\gamma]$, . . . are necessary when the weight of a linear function of the unknowns is required, as will be seen presently. (See Art. 101.)

It is evident from the form of the weight equations that if the elimination is carried through by the method of substitution, the successive steps to the left of the sign of equality are the same as in Art. 88. Hence if the equations are arranged so that the unknown whose weight is required—z, for example—is found first, we should have the forms

x	y	z	
$[aa]$	$+[ab]$	$+[ac]$	$[al]$
	$+[bb.1]$	$+[bc.1]$	$[bl.1]$
		$+[cc.2]$	$[cl.2]$

$$\therefore [cc.2]z = [cl.2]$$

$[\alpha\gamma]$	$[\beta\gamma]$	$[\gamma\gamma]$	
$[aa]$	$+[ab]$	$+[ac]$	0
	$+[bb.1]$	$+[bc.1]$	0
		$+[cc.2]$	1

$$\therefore [cc.2][\gamma\gamma] = 1$$

Hence the coefficient of the unknown first found in the ordinary solution of the normal equations is the weight of that unknown. By a separate elimination for each unknown the weight of that unknown could be found as above, but the process would be intolerably tedious.

95. *Special Cases of Two and Three Unknowns.*—We may, however, from the preceding derive formulas for the weights

in a series of normal equations containing not more than three unknowns, which are easy of application.

Thus with two unknowns, x and y, y being found first,

$$p_y = [bb.1]$$
$$= [bb] - \frac{[ab]^2}{[aa]}$$

In the reverse order, x being found first,

$$p_x = [aa] - \frac{[ab]^2}{[bb]}$$

or

$$p_x = \frac{\lambda}{[bb]},\ p_y = \frac{\lambda}{[aa]}$$

where

$$\lambda = [aa][bb] - [ab][ab]$$

With three unknowns, x, y, z, performing the elimination of the normal equations in the order z, y, x, we have

$$p_z = [cc.2]$$

$$p_y = \frac{[bb.1][cc.2]}{[cc.1]}$$

$$p_x = \frac{[aa][bb.1][cc.2]}{[bb][cc] - [bc][bc]}$$

which expressions are easily transformed into

$$p_z = \frac{\lambda}{[aa][bb] - [ab]^2}$$

$$p_y = \frac{\lambda}{[aa][cc] - [ac]^2}$$

$$p_x = \frac{\lambda}{[bb][cc] - [bc]^2}$$

where

$$\lambda = [aa][bb][cc] + 2[ab][bc][ac] - [aa][bc]^2 - [bb][ac]^2 - [cc][ab]^2$$

From these formulas the weights of the unknowns can be found directly without solving the normal equations. If the normal equations have simple coefficients it is much more rapid to find the weights in this way and solve the equations by ordinary algebra rather than by the Gaussian method. But when the number of unknowns exceeds three this becomes too cumbersome.

Ex. To find the weights of the adjusted angles in Ex. 4, Art. 83. Here

$$\begin{aligned}\lambda &= 12 \times 11 \times 15 - 12 \times 16 - 11 \times 49 \\ &= 1249\end{aligned}$$

and

$$p_x = \frac{1249}{149} = 8.4$$

$$p_y = \frac{1249}{131} = 9.5$$

$$p_z = \frac{1249}{132} = 9.5$$

If u_x, u_y, u_z denote the reciprocals of p_x, p_y, p_z respectively, then

$$\begin{aligned}u_x &= 0.1193 \\ u_y &= 0.1049 \\ u_z &= 0.1057\end{aligned}$$

96. *Modification of General Method.*—To carry out the method of Art. 94 directly as stated would be excessively troublesome, and various modifications have been proposed. The following scheme, which consists in running the weight equations together, will be found very convenient.

Take, for simplicity in writing, three unknowns, x, y, z, and to the ordinary form of the normal equations as arranged for solution add the columns

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

the check being carried throughout.

Perform the elimination exactly as stated in Art 88, and find the values of the unknowns in the usual way. We have then

x	y	z		R	S	T	Check.
$[aa]$ $[ab]$ $[ac]$	$[ab]$ $[bb]$ $[bc]$	$[ac]$ $[bc]$ $[cc]$	$[al]$ $[bl]$ $[cl]$	1 0 0	0 1 0	0 0 1	$[as] + 1$ $[bs] + 1$ $[cs] + 1$
1	$\frac{[ab]}{[aa]}$	$\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$	$\frac{1}{[aa]}$			
	$[bb.1]$ $[bc.1]$	$[bc.1]$ $[cc\ 1]$	$[bl.1]$ $[cl.1]$	R_1 $-\frac{[ac]}{[aa]}$	1 0	1	
	1	$\frac{[bc\ 1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$	$\frac{R_1}{[bb.1]}$	$\frac{1}{[bb.1]}$		
		$[cc.2]$	$[cl.2]$	R_2	S_2	1	
		1	$\frac{[cl.2]}{[cc.2]}$	$\frac{R_2}{[cc.2]}$	$\frac{S_2}{[cc.2]}$	$\frac{1}{[cc.2]}$	

where

$$0 = \frac{[ab]}{[aa]} + R_1$$

$$0 = \frac{[ac]}{[aa]} + \frac{[bc.1]}{[bb.1]} R_1 + R_2$$

$$0 = \frac{[bc.1]}{[bb.1]} + S_2$$

Now, taking the first column in the table under the heading R, and attending to Art. 94, we have

$$[\alpha\gamma] = \frac{R_2}{[cc.2]}$$

$$[\alpha\beta] = \frac{R_1}{[bb.1]} - \frac{[bc.1]}{[bb.1]}[\alpha\gamma]$$

$$= \frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]}$$

$$u_x = [aa] = \frac{1}{[aa]} - \frac{[ab]}{[aa]}[\alpha\beta] - \frac{[ac]}{[aa]}[\alpha\gamma]$$

$$= \frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]}$$

Similarly for the column under S,

$$[\beta\gamma] = \frac{S_2}{[cc.2]}$$

$$u_y = [\beta\beta] = \frac{1}{[bb.1]} + \frac{S_2^2}{[cc.2]}$$

and for the column under T,

$$u_z = [\gamma\gamma] = \frac{1}{[cc.2]}$$

Also it is evident that

$$z = \frac{[cl.2]}{[cc.2]}$$

$$y = \frac{[bl.1]}{[bb.1]} + \frac{[cl.2]}{[cc.2]}S_2$$

$$x = \frac{[al]}{[aa]} + \frac{[bl.1]}{[bb.1]}R_1 + \frac{[cl.2]}{[cc.2]}R_2$$

The forms of the expressions for $[\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$, . . . show that these quantities may be conveniently computed from the preceding tabular elimination scheme. Thus the sum of the products of each pair of numbers bracketed under the heads R, S, T will give u_x, u_y, u_z respectively.

The convenience of this form is seen in such a case as the following, which is of common occurrence. In a set of, say, 40 normal equations the weights of 10 of the unknowns may be required. These 10 would be placed last in the solution of the equations, and the extra columns R, S, . . . added after 30 of the unknowns had been eliminated, thus giving the weights required, with a trifling increase of work.

Ex. 1. Given the normal equations

$$\begin{aligned} 12x \qquad\quad - \;\; 7z &= R \\ +\, 11y - \;\; 4z &= S \\ -\;\; 7x - \;\; 4y + 15z &= T\,. \end{aligned}$$

to find the weights of y and z.

x	y	z	S	T
+ 12	0	− 7		
	+ 11	− 4		
− 7	− 4	+ 15		
1	0	− 0.5833		
	+ 11	− 4	+ 1	
	− 4	+ 10 9169		+ 1
	1	− 0.3636	+ 0.0909	
		+ 9.4625	+ 0.3636	+ 1
		1	+ 0.0384	+ 0.1057

Hence

$$\begin{aligned} u_z &= 1 \times 0.1057 &&= 0.1057 \\ u_y &= 1 \times 0.0909 + 0.3636 \times 0.0384 &&= 0.1049 \end{aligned}$$

agreeing with the values in Art. 95.

Ex. 2. Show that

$$\begin{aligned} [bl.1] &= [al]R_1 + [bl] \\ [cl.2] &= [al]R_2 + [bl]S_2 + [cl] \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Ex. 3. Show that the multipliers R_1, R_2, . . . satisfy the conditions

$$[aa]R_1 + [ab] = 0$$

$$\begin{aligned} [aa]R_2 + [ab]S_2 + [ac] &= 0 \\ [ab]R_2 + [bb]S_2 + [bc] &= 0 \end{aligned}$$

$$\begin{aligned} [aa]R_3 + [ab]S_3 + [ac]T_3 + [ad] &= 0 \\ [ab]R_3 + [bb]S_3 + [bc]T_3 + [bd] &= 0 \\ [ac]R_3 + [bc]S_3 + [cc]T_3 + [cd] &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned}$$

97. Second Method of Finding the Weights of the Unknowns.—If we multiply the first of the normal equations 1, Art. 93, by $[\alpha\alpha]$, the second by $[\alpha\beta]$, the third by $[\alpha\gamma]$, and so on; add the products, and attend to equations 10, Art. 93, we obtain

$$x = [\alpha\alpha][al] + [\alpha\beta][bl] + [\alpha\gamma][cl] + \ldots$$

Similarly

$$y = [\alpha\beta][al] + [\beta\beta][bl] + [\beta\gamma][cl] + \ldots$$
$$z = [\alpha\gamma][al] + [\beta\gamma][bl] + [\gamma\gamma][cl] + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence since $[\alpha\alpha]$, $[\beta\beta]$, . . . are the reciprocals of the weights of x, y, . . ., this method of finding the weights may be stated as follows:

In any given series of observations, having formed the normal equations, replace the numerical absolute terms by the general symbols $[al]$, $[bl]$, . . . and find by any method of elimination the values of x, y, . . ., in terms of $[al]$, $[bl]$, . . .; then the weight of x is the reciprocal of the coefficient of $[al]$ in the value of x, the weight of y is the reciprocal of the coefficient of $[bl]$ in the value of y, and so on.

The coefficients of the remaining symbols for the absolute terms in the expressions for x, y, . . . give the values of $[\alpha\beta]$, $[\alpha\gamma]$, . . .; $[\beta\gamma]$, . . .; . . . and the numerical values of the unknowns x, y, . . . may be found by substituting for $[al]$, $[bl]$, . . . their numerical values.

In this method of computing the values of the unknowns and their weights a machine can be used with great advantage.

The formulas of Art. 96 are easily derived from the preceding principles. For solving the normal equations

$$[aa]x + [ab]y + [ac]z = [al]$$
$$[ab]x + [bb]y + [bc]z = [bl]$$
$$[ac]x + [bc]y + [cc]z = [cl]$$

by the method of substitution we have for the first unknown

$$x = \frac{[al]}{[aa]} + \frac{[bl.1]}{[bb.1]} R_1 + \frac{[cl.2]}{[cc.2]} R_2$$

$$= [al]\left(\frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]}\right)$$

$$+ [bl]\left(\frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]}\right) + [cl]\frac{R_2}{[cc.2]}$$

Comparing this with the general expression for x in Eq. 1,

$$[\alpha\alpha] = \frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]}$$

$$[\alpha\beta] = \frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]}$$

$$[\alpha\gamma] = \frac{R_2}{[cc.2]}$$

Similarly for y and z.

Ex. To find the weights of the unknowns in Ex. 4, Art. 83. Solving the normal equations in general terms,

$$x = 0.1193\,[al] + 0.0224\,[bl] + 0.0616\,[cl]$$
$$y = 0.0224\,[al] + 0.1049\,[bl] + 0.0384\,[cl]$$
$$z = 0.0616\,[al] + 0.0384\,[bl] + 0.1057\,[cl]$$

Hence

$$u_x = [\alpha\alpha] = 0.1193$$
$$u_y = [\beta\beta] = 0.1049$$
$$u_z = [\gamma\gamma] = 0.1057$$

as found in Art. 95.

98. In deducing the formulas for the precision of the adjusted values in a series of normal equations we have, for convenience in writing, taken the observation equations to be reduced to weight unity, and the normal equations, consequently, to be of the form

$$\begin{aligned}
[aa]x + [ab]y + \ldots &= [al] \\
[ab]x + [bb]y + \ldots &= [bl] \qquad (1) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot
\end{aligned}$$

The formulas with the weight symbols introduced, corresponding to those found in the preceding articles, are easily derived from them by writing $a\sqrt{p}$, $b\sqrt{p}$, . . . $l\sqrt{p}$, for a, b, . . . l, and $\alpha\sqrt{u}$, $\beta\sqrt{u}$, . . . for α, β, . . . respectively. (See Art. 58.)

Thus, for example, from the normal equations

$$\begin{aligned} [paa]x + [pab]y + \ldots &= [pal] \\ [pab]x + [pbb]y + \ldots &= [pbl] \\ \cdots\cdots\cdots\cdots \end{aligned} \qquad (2)$$

we should have

$$\begin{aligned} x = [al] &= [uaa][pal] + [u\alpha\beta][pbl] + \ldots \\ y = [\beta l] &= [u\alpha\beta][pal] + [u\beta\beta][pbl] + \ldots \\ \cdots\cdots\cdots\cdots \end{aligned} \qquad (3)$$

and by equating coefficients of l_1, l_2, . . . in the first expression,

$$\begin{aligned} [u\alpha\alpha]a_1 + [u\alpha\beta]b_1 + \ldots &= u_1\alpha_1 \\ [u\alpha\beta]a_1 + [u\beta\beta]b_1 + \ldots &= u_1\beta_1 \\ \cdots\cdots\cdots\cdots \end{aligned} \qquad (4)$$

99. **To Find the m. s. e. μ of a Single Observation.**—If the errors Δ were known—that is, if the n observation equations were

$$\begin{aligned} a_1x_0 + b_1y_0 + \ldots - l_1 &= \Delta_1 \\ a_2x_0 + b_2y_0 + \ldots - l_2 &= \Delta_2 \\ \cdots\cdots\cdots\cdots \end{aligned} \qquad (1)$$

where x_0, y_0, . . . are the true values of the unknowns—we should have at once

$$\mu^2 = \frac{[\Delta\Delta]}{n}$$

But we have only the residuals v with the observation equations

$$\begin{aligned} a_1x + b_1y + \ldots - l_1 &= v_1 \\ a_2x + b_2y + \ldots - l_2 &= v_2 \\ \cdots\cdots\cdots\cdots \end{aligned} \qquad (2)$$

where x, y, . . . are the most probable values of the unknowns. We must, therefore, express $[\varDelta\varDelta]$ in terms of the residuals v in order to find μ.

From the two sets of equations, by subtracting in pairs,

$$\begin{aligned}\varDelta_1 &= v_1 + a_1(x_0 - x) + b_1(y_0 - y) + \ldots \\ \varDelta_2 &= v_2 + a_2(x_0 - x) + b_2(y_0 - y) + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned} \qquad (3)$$

Now, taking the m. s. e. μ_x, μ_y, . . . to be the errors of x, y, . . . , that is, to be equal to $x_0 - x$, $y_0 - y$, . . . , we have from Eq. 4, Art. 93,

$$x_0 - x = \mu\sqrt{[\alpha\alpha]}, \quad y_0 - y = \mu\sqrt{[\beta\beta]}, \ \ldots$$

and therefore

$$\begin{aligned}\varDelta_1 &= v_1 + \mu(a_1\sqrt{[\alpha\alpha]} + b_1\sqrt{[\beta\beta]} + \ldots) \\ \varDelta_2 &= v_2 + \mu(a_2\sqrt{[\alpha\alpha]} + b_2\sqrt{[\beta\beta]} + \ldots) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Squaring, adding, and attending to equations 10, Art. 93, we have approximately, n_i being the number of unknowns,

$$[\varDelta\varDelta] = [vv] + n_i\mu^2 \qquad (4)$$

Putting $[\varDelta\varDelta] = n\mu^2$, there results

$$\mu^2 = \frac{[vv]}{n - n_i} \qquad (5)$$

the expression required.

Reasoning as in Peters' formula, Art. 47, we easily deduce from (4)

$$\mu = 1.2533\frac{[v]}{\sqrt{n(n-n_i)}} \qquad (6)$$

which is known as Lüroth's formula (*Astron. Nachr.*, 1740).

When $n_i = 1$, equations 5 and 6 reduce to Bessel's and Peters' formulas respectively (Arts. 43, 47).

100. *Methods of Computing* $[vv]$.—(a) The ordinary method is to substitute the values of the unknowns found from the solution of the normal equations in the observation equa-

tions, and thence find $v_1, v_2, \ldots$ The sum of the squares of these residuals will give $[vv]$.

The residuals having to be found, for the purpose of testing the quality of the work this method of computing $[vv]$ is on the whole as short as any.

As checks on the values of $[vv]$ found in this way the following are of value:

(b) If we multiply each observation equation by its v and take the sum of the products, then, remembering that $[av] = 0$, $[bv] = 0$, . . ., we find

$$[vv] = -[vl]$$

(c) If we multiply each of the observation equations by its l and take the sum of the products,

$$\begin{aligned}[al]x + [bl]y + \ldots - [ll] &= [vl] \\ &= -[vv]\end{aligned}$$

(d) We have for two unknowns, x and y,

$$\begin{aligned}
[vv] &= [(ax + by - l)^2] \\
&= [aa]x^2 + 2[ab]xy + [bb]y^2 - 2[al]x - 2[bl]y + [ll] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + \left([bb] - \frac{[ab]}{[aa]}[ab]\right)y^2 \\
&\qquad - \left(2[bl] - 2\frac{[ab]}{[aa]}[al]\right)y + [ll] - \frac{[al]}{[aa]}[al] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]y^2 - 2[bl.1]y + [ll.1] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]\left(y - \frac{[bl.1]}{[bb.1]}\right)^2 \\
&\qquad + [ll.1] - \frac{[bl.1]}{[bb.1]}[bl.1] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]\left(y - \frac{[bl.1]}{[bb.1]}\right)^2 + [ll.2]
\end{aligned}$$

and generally for m unknowns

$$
\begin{aligned}
[vv] &= [(ax + by + \ldots - l)^2] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y + \ldots - \frac{[al]}{[aa]}\right)^2 \\
&\quad + [bb.1]\left(y + \frac{[bc.1]}{[bb.1]}z + \ldots - \frac{[bl.1]}{[bb.1]}\right)^2 + \ldots + [ll.m]
\end{aligned}
$$

Now, from (9), Art. 86, the coefficients of $[aa]$, $[bb.1]$, . . . are each equal to zero. Hence

$$
\begin{aligned}
[vv] &= [ll.m] \\
&= [ll] - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \frac{[cl.2]^2}{[cc.2]} - \ldots
\end{aligned}
$$

This expression was first given by Gauss (*De Elementis Ellipticis Palladis*, Art. 13). Its form suggests that if we add an extra column to the normal equations, as shown in the following scheme, we shall find $[vv]$ at the same time as the first unknown. This is analytically very elegant, and, as the check (see Art. 87) can be carried with this column through the solution of the normal equations, it may be used for finding $[vv]$, if one is computing alone. Only one extra term $[ll]$ has to be computed while forming the normal equations.

The scheme is as follows:

x	y	z	
$[aa]$	$[ab]$ $[bb]$	$[ac]$ $[bc]$ $[cc]$	$[al]$ $[bl]$ $[cl]$ $[ll]$
1	$\frac{[ab]}{[aa]}$ $[bb.1]$	$\frac{[ac]}{[aa]}$ $[bc.1]$ $[cc.1]$	$\frac{[al]}{[aa]}$ $[bl.1]$ $[cl.1]$ $[ll.1] = [ll] - \frac{[al]}{[aa]}[al]$
	1	$\frac{[bc.1]}{[bb.1]}$ $[cc.2]$	$\frac{[bl.1]}{[bb.1]}$ $[cl.2]$ $[ll.2] = [ll.1] - \frac{[bl.1]}{[bb.1]}[bl.1]$.

Ex. 1. To find the m. s. e. of the adjusted values of the unknowns found in Ex. 4, Art. 83.

The first step is to find $[pvv]$. This we shall do in the four ways indicated.

(a)

v	p	pvv
-0.05	5	0.01
-0.36	7	0.91
$+0.68$	4	1.85
-0.03	7	0.01
-0.62	4	1.54
		$4.32 = [pvv]$

(b)

$$
\begin{aligned}
p_1l_1v_1 &= 0 \\
p_2l_2v_2 &= 0 \\
p_3l_3v_3 &= 0 \\
p_4l_4v_4 &= 7 \times 0.76 \times -0.03 = -0.16 \\
p_5l_5v_5 &= 4 \times 1.66 \times -0.62 = \underline{-4.12} \\
&\qquad -4.28 = +[pvl] = -[pvv]
\end{aligned}
$$

(c)

$$
\begin{aligned}
[pal]x &= -\ \ 5.32 \times -0.05 = 0.27 \\
[pbl]y &= -\ \ 6.64 \times -0.36 = 2.39 \\
[pcl]z &= +11.96 \times +0.68 = \underline{8.13} \qquad 10.79 \\
p_1l_1l_1 &= 0 \\
p_2l_2l_2 &= 0 \\
p_3l_3l_3 &= 0 \\
p_4l_4l_4 &= 7 \times (0.76)^2 \qquad = \ \ 4.04 \\
p_5l_5l_5 &= 4 \times (1.66)^2 \qquad = \underline{11.02} \qquad \underline{15.06} \\
&\qquad\qquad 4.27 = [pvv]
\end{aligned}
$$

(d) We find $[pll] = 15.06$.

The solution of the normal equations, with the extra column for $[pll]$ added, would be, according to the foregoing scheme,

x	y	z		
12	0	-7	-5.32	
	$+11$	-4	-6.64	
		$+15$	$+11.96$	
			$+15.06$	2.36
1	0	-0.583	-0.443	
	$+11$	-4	-6.64	
		$+10.917$	$+8.857$	
			$+12.70$	$(=15.06-2.36)$
	1	-0.364	-0.604	
		$+9.462$	$+6.422$	
			$+8.69$	$(=12.70-4.01)$
		1	$+0.68$	
		$[pvv]$	$+4.30$	$(=8.69-4.39)$

Mean value of $[pvv] = 4.29$

Hence (Art. 99)

$$\mu = \sqrt{\frac{4.29}{5-3}} = 1''.47$$

and (see Ex., Art. 97)

$$\mu_x = 1''.47\sqrt{0.1193} = 0''.51, \qquad \mu_y = 1''.47\sqrt{0.1040} = 0''.48, \qquad \mu_z = 1''.47\sqrt{0.1057} = 0''.48$$

Ex. 2. Show that

$$[\Delta v] = [vv]$$

[Multiply equations 1, Art. 91, by $v_1, v_2, \ldots$ and add. Then since

$$[av] = 0, \ [bv] = 0, \ \ldots$$
$$\therefore [\Delta v] = -[lv] \quad]$$

Ex. 3. Show that

$$[vv]=[\Delta\Delta]-\frac{[a\Delta]^2}{[aa]}-\frac{[b\Delta.1]^2}{[bb.1]}-\ldots$$

[Form the normal equations from equations 3, Art. 99, and

$$\begin{aligned}[aa](x_0-x)+[ab](y_0-y)+\ldots&=[a\Delta]\\ [ab](x_0-x)+[bb](y_0-y)+\ldots&=[b\Delta]\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}$$

since $[av]=[bv]=\ldots=0$

Hence from Art. 100

$$[\Delta\Delta]=[vv]+\frac{[a\Delta]^2}{[aa]}+\frac{[b\Delta.1]^2}{[bb.1]}+\ldots\ ]$$

Ex. 4. From the equation

$$[al]x+[bl]y+\ldots-[ll]=-[vv]$$

and

$$\begin{aligned}x&=\frac{[al]}{[aa]}+\frac{[bl.1]}{[bb.1]}R_1+\frac{[cl.2]}{[cc.2]}R_2+\ldots\\ y&=\qquad\ \ \frac{[bl.1]}{[bb.1]}\qquad+\frac{[cl.2]}{[cc.2]}S_2+\ldots\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}$$

deduce

$$[vv]=[ll]-\frac{[al]^2}{[aa]}-\frac{[bl.1]^2}{[bb.1]}-\ldots$$

Ex. 5. Prove that $[\alpha v]=[\beta v]=\ldots=0$

Ex. 6. From

$$[vv]=[\Delta\Delta]-\frac{[a\Delta]^2}{[aa]}-\frac{[b\Delta.1]^2}{[bb.1]}-\ldots$$

deduce the formula

$$\mu^2=\frac{[vv]}{n-n_i}$$

101. **To Find the Precision of any Function of the Adjusted Values X, Y, . . .**—This is the more general case of the problem just discussed. The method of solution is:

First, to find μ, the m. s. e. of an observation of weight unity, and next p_F the weight of the function, whence the m. s. e. of the function is given by

$$\mu\sqrt{u_F}$$

where u_F is the reciprocal of p_F.

The value of μ is computed from (5) or (6), Art. 99.

Next, to find u_F. Let the function be

$$F = f(X, Y, \ldots) \qquad (1)$$

in which X, Y, . . . are functions of the independently observed quantities M_1, M_2, . . . M_n.

By differentiation

$$dF = \frac{\partial F}{\partial M_1}dM_1 + \frac{\partial F}{\partial M_2}dM_2 + \ldots + \frac{\partial F}{\partial M_n}dM_n$$

and therefore, since M_1, M_2, . . . are independent,

$$u_F = \left(\frac{\partial F}{\partial M_1}\right)^2 u_1 + \left(\frac{\partial F}{\partial M_2}\right)^2 u_2 + \ldots + \left(\frac{\partial F}{\partial M_n}\right)^2 u_n \quad (2)$$

where u_1, u_2, . . . u_n are the reciprocals of the weights of the observed quantities.

(a) Instead, however, of using this general formula directly, it is, in general, a great saving of labor to compute from a modified form as follows, by means of which much of the work already done in solving the normal equations may be utilized.

Reducing the function to the linear form, we have, adopting the notation of Art. 81,

$$F = f(X'+x,\ Y'+y,\ \ldots)$$
$$= f(X',\ Y',\ \ldots) + \frac{\partial F}{\partial X'}x + \frac{\partial F}{\partial Y'}y + \ldots$$

or, as it may be written,

$$dF = G_1x + G_2y + \ldots \tag{3}$$

Now, since $x, y, \ldots$ are not independent, but are connected by the equations

$$[aa]x + [ab]y + \ldots = [al]$$
$$[ab]x + [bb]y + \ldots = [bl]$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

we must get rid of this entanglement by expressing these quantities $x, y, \ldots$ in terms of $l_1, l_2, \ldots$, which are independent of each other. From Art. 93 we may write

$$x = [\alpha l],\ y = [\beta l],\ \ldots$$

where $\alpha_1, \alpha_2, \ldots$; $\beta_1, \beta_2, \ldots$; $\ldots$ are functions of $a_1, b_1, \ldots$; $a_2, b_2, \ldots$; $\ldots$ Hence, substituting in (3)

$$dF = (G_1\alpha_1 + G_2\beta_1 + \ldots)l_1 + (G_1\alpha_2 + G_2\beta_2 + \ldots)l_2 + \ldots$$

and, therefore (Arts. 19, 56), since the observation equations have been reduced to weight unity,

$$\begin{aligned} u_F &= (G_1\alpha_1 + G_2\beta_1 + \ldots)^2 + (G_1\alpha_2 + G_2\beta_2 + \ldots)^2 + \ldots \\ &= G_1^2[\alpha\alpha] + 2G_1G_2[\alpha\beta] + \ldots \\ &\qquad\quad + G_2^2[\beta\beta] + \ldots \\ &\qquad\qquad\qquad + \ldots \end{aligned} \tag{4}$$

where $[\alpha\alpha]$, $[\alpha\beta] \ldots$ may be found in the manner indicated in Arts. 94, 96, or 97. Hence U is known.

(b) Eq. 4 may be written

$$u_F = G_1Q_1 + G_2Q_2 + \ldots + G_nQ_n \tag{5}$$

where

$$\begin{aligned} Q_1 &= [a\alpha]G_1 + [\alpha\beta]G_2 + \dots \\ Q_2 &= [\alpha\beta]G_1 + [\beta\beta]G_2 + \dots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{6}$$

that is, where (see Eq. 1, Art. 97)

$$\begin{aligned} G_1 &= [aa]Q_1 + [ab]Q_1 + \dots \\ G_2 &= [ab]Q_2 + [bb]Q_2 + \dots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{7}$$

Hence the weight of a function

$$G_1 x + G_2 y + \dots$$

of several independent unknowns $x, y, \dots$ is found from

$$u_F = [GQ]$$

where $Q_1, Q_2, \dots$ satisfy the equations

$$\begin{aligned} [aa]Q_1 + [ab]Q_2 + \dots &= G_1 \\ [ab]Q_1 + [bb]Q_2 + \dots &= G_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned}$$

Therefore we conclude that, *if in a series of observation equations the values of the unknowns $x, y, \dots$ are required, as well as their weights or the weight of any function of them, these results can be found at one time by making a solution of the normal equations for finding $x, y, \dots$ in general terms, and then substituting for $[al]$, $[bl]$, $\dots$ their numerical values on the one hand and the values of G_1, G_2, $\dots$ on the other.*

(c) This result may be stated in other forms. Thus from Eq. 4, by substituting for $[\alpha\alpha]$, $[\alpha\beta]$, $\dots$ their values from Art. 96, or by substituting for Q_1, Q_2, $\dots$ their values in (6) as expressed in Art. 96, we have, after a simple reduction,

$$u_F = \frac{G_1^2}{[aa]} + \frac{(G_1 R_1 + G_2)^2}{[bb.1]} + \frac{(G_1 R_2 + G_2 S_2 + G_3)^2}{[cc.2]} + \dots \tag{8}$$

Comparing this expression with (d), Art. 100, it is evident that the several terms are such as would result from the

following elimination (Ex. three unknowns) by finding the products of the quantities bracketed:

x	y	z	
$[aa]$ $[ab]$ $[ac]$	$[ab]$ $[bb]$ $[bc]$	$[ac]$ $[bc]$ $[cc]$	G_1 G_2 G_3
1	$\frac{[ab]}{[aa]}$ $[bb.1]$ $[bc.1]$	$\frac{[ac]}{[aa]}$ $[bc.1]$ $[cc.1]$	$\frac{G_1}{[aa]}$ $G_1R_1 + G_2$
	1	$\frac{[bc.1]}{[bb.1]}$ $[cc.2]$	$\frac{G_1R_1 + G_2}{[bb.1]}$ $G_1R_2 + G_2S_2 + G_3$
		1	$\frac{G_1R_2 + G_2S_2 + G_3}{[cc.2]}$

(d) The expression (8) for u_F may be easily transformed into

$$k_0^2[aa] + k_1^2[bb.1] + k_2^2[cc.2] + \ldots$$

where

$$\begin{aligned} &[aa]k_0 &&= -G_1 \\ &[ab]k_0 + [bb.1]k_1 &&= -G_2 \\ &[ac]k_0 + [bc.1]k_1 + [cc.2]k_2 &&= -G_3 \\ &\ldots \quad \ldots \quad \ldots && \ldots \end{aligned}$$

Circumstances must decide which of the four forms given should be chosen in any special case. A machine can be used to the best advantage with the second and third forms. The third form is also convenient when the weights alone are required, without the values of the unknowns, and the second when the values of the unknowns can be found by an easier method of solution than the Gaussian method of substitution.

Ex. 1. In Ex. 4, Art. 83, it is required to find the m. s. e. of the angle *PSB*.

The function is

$$dF = -x + z$$

First Solution. From Eq. 4,

$$\begin{aligned} u_F &= [\alpha\alpha] - 2[\alpha\gamma] + [\gamma\gamma] \\ &= 0.1193 - 2 \times 0.0616 + 0.1057 \text{ (from Ex., Art. 97).} \\ &= 0.1018 \end{aligned}$$

Second Solution. From equations 7,

$$\begin{aligned} +12Q_1 \qquad\qquad - \ 7Q_3 &= -1 \\ +11Q_2 - \ 4Q_3 &= \ \ 0 \\ - \ 7Q_1 - \ 4Q_2 + 15Q_3 &= +1 \end{aligned}$$

Hence

$$Q_1 = -0.0577, \quad Q_3 = +0.0440$$

and

$$\begin{aligned} u_F &= -1 \times -0.0577 + 1 \times 0.0440 \\ &= 0.1017 \end{aligned}$$

Third Solution. Add the extra column G to the solution of the normal equations, which would give the scheme

x	y	z	G
+12		−7	−1
	+11	−4	0
		+15	+1
+1		−0.5833	−0.0833
	+11	−4.	0
		+10.9169	+0.4169
	+1	−0.3636	0
		+9.4625	+0.4169
		+1	+0.0441

Hence

$$\begin{aligned} u_F &= -1 \times -0.0833 + 0.4169 \times 0.0441 \\ &= 0.1017 \end{aligned}$$

Fourth Solution.

$$
\begin{aligned}
12k_0 \qquad\qquad\qquad &= +1 \\
11k_1 \qquad\quad &= 0 \\
-7k_0 - 4k_1 + 9.4625k_2 &= -1 \\
\therefore\ k_0 = 0.0833,\ k_1 = 0,\ k_2 &= -0.0440 \\
u_F = (0.0833)^2 \times 12 + (0.0440)^2 &\times 9.4625 \\
= 0.1016 &
\end{aligned}
$$

Also

$$
\begin{aligned}
\mu_F &= 1''.47\sqrt{0.102} \\
&= 0''.47
\end{aligned}
$$

Ex. 2. Given the observation equations

$$
\begin{aligned}
a_1x + b_1y &= l_1 \\
a_2x + b_2y &= l_2 \\
\cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\
a_nx + b_ny &= l_n
\end{aligned}
$$

to find the weight of $fx + gy$.

[The normal equations are

$$
\begin{aligned}
[aa]x + [ab]y &= [al] \\
[ab]x + [bb]y &= [bl]
\end{aligned}
$$

$$
\therefore\ u_F = \frac{1}{[aa][bb.1]}\{[bb]f^2 - 2[ab]fg + [aa]g^2\}\quad]
$$

102. *To find the average value of the ratio of the weight of the observed value of a quantity to that of its adjusted value in a system of independently observed quantities.*

The adjusted value of the first observed quantity M_1 is $M_1 + v_1$. From Art. 81 it follows that the weight of $M_1 + v_1$ is the same as the weight of $l_1 + v_1$. Now,

$$
l_1 + v_1 = a_1x + b_1y + \ldots \tag{1}
$$

Hence if P_1 is the weight of the adjusted value $M_1 + v_1$, that is, is the weight of the function $a_1x + b_1y + \ldots$, and $p_1, p_2, \ldots$ are the weights of $l_1, l_2, \ldots$, we have

$$
\frac{1}{P_1} = a_1Q_1 + b_1Q_2 + \ldots \tag{2}
$$

where (see Eq. 4, Art. 98, (b) Art. 101)

$$
\begin{aligned}
Q_1 &= [u\alpha\alpha]a_1 + [u\alpha\beta]b_1 + \ldots = u_1\alpha_1 \\
Q_2 &= [u\alpha\beta]a_1 + [u\beta\beta]b_1 + \ldots = u_1\beta_1 \\
\cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{3}
$$

Therefore by substitution of Q_1, Q_2, . . . in (2),

$$\frac{p_1}{P_1} = a_1\alpha_1 + b_1\beta_1 + \ldots$$

Similarly

$$\frac{p_2}{P_2} = a_2\alpha_2 + b_2\beta_2 + \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence by addition

$$\left[\frac{p}{P}\right] = [a\alpha] + [b\beta] + \ldots \text{ to } n_i \text{ terms}$$

$$= n_i, \text{ the number of independent unknowns.}$$

This result may be very readily derived directly as follows: In (1) put $[\alpha l]$ for x, $[\beta l]$ for y, . . . , and we have

$$l_1 + v_1 = (a_1\alpha_1 + b_1\beta_1 + \ldots)l_1 + (a_1\alpha_2 + b_1\beta_2 + \ldots)l_2 + \ldots$$

Hence, since l_1, l_2, . . . are independent,

$$\frac{1}{P_1} = (a_1\alpha_1 + b_1\beta_1 + \ldots)^2\frac{1}{p_1} + (a_1\alpha_2 + b_1\beta_2 + \ldots)^2\frac{1}{p_2} + \ldots$$

$$= a_1\{[u\alpha\alpha]a_1 + [u\alpha\beta]b_1 + \ldots\}$$

$$+ b_1\{[u\alpha\beta]a_1 + [u\beta\beta]b_1 + \ldots\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= u_1(a_1\alpha_1 + b_1\beta_1 + \ldots) \tag{4}$$

as before.

Ex. To check the weights of the adjusted values of the angles found in Ex. 4, Art. 83.

The weights of the measured values of the angles are

5, 7, 4, 7, 4

The weights of the adjusted values are (Ex., Art. 95; Ex. 1, Art. 101)

8.4, 9.5, 9.5, 9.8, 7.5

Also

$$\frac{5}{8.4} + \frac{7}{9.5} + \frac{4}{9.5} + \frac{7}{9.8} + \frac{4}{7.5}$$

$$= 3$$

$$= \text{the number of independent unknowns,}$$

as it should.

103. From the principle just proved we may derive a proof of the formula found in Art. 99 for the m. s. e. of an observation of weight unity in a series of n observations involving n_i independent unknowns. It is analogous to the proof given for the m. s. e. of a single observation in a series of n observations of one unknown.

The errors of the observed values $M_1, M_2, \ldots M_n$ are $\Delta_1, \Delta_2, \ldots \Delta_n$, and the errors of the most probable values $V_1, V_2, \ldots V_n$ may be assumed to be $\mu_{V_1}, \mu_{V_2}, \ldots \mu_{V_n}$ respectively. Hence since

$$M = V - v$$

we have

$$\begin{aligned} \Delta_1 &= \mu_{V_1} - v_1 \\ \Delta_2 &= \mu_{V_2} - v_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \Delta_n &= \mu_{V_n} - v_n \end{aligned}$$

and, therefore,

$$[p\Delta\Delta] = [p\mu_V\mu_V] + [pvv] \tag{1}$$

Again, since P_1 is the weight of V_1,

$$\mu_{V_1}^2 = \frac{\mu^2}{P_1}$$

where μ is the m. s. e. of an observation of weight unity. Hence

$$\begin{aligned} p_1 \mu_{V_1}^2 &= \frac{p_1}{P_1}\mu^2 \\ p_2 \mu_{V_2}^2 &= \frac{p_2}{P_2}\mu^2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ p_n \mu_{V_n}^2 &= \frac{p_n}{P_n}\mu^2 \end{aligned}$$

By addition,

$$\begin{aligned} [p\mu_V\mu_V] &= \left[\frac{p}{P}\right]\mu^2 \\ &= n_i\mu^2 \end{aligned} \tag{2}$$

But by definition

$$\mu^2 = \frac{[p\Delta\Delta]}{n}$$

Substituting in (1), we have finally

$$\mu^2 = \frac{[pvv]}{n - n_i}$$

the result required.

Miscellaneous Examples and Artifices of Elimination.

In this section will be discussed several problems of importance, and also certain artifices of elimination which may often be employed with advantage in the solution of observation equations.

104. The labor of solving and finding the values of the unknowns may often be shortened by taking advantage of some principle inherent in the form of the observation equations themselves. For example, if we have a series of observation equations containing two unknowns, and of which the coefficient of the first unknown is unity, instead of solving in the usual way we may reduce the observation equations to equations containing the second unknown only, and thus solve more readily.

Given

$$\begin{array}{ll} x + b_1 y = l_1 & \text{weight } p_1 \\ x + b_2 y = l_2 & \quad\text{“} \quad p_2 \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \end{array}$$

Forming the normal equations in the usual way, we have

$$\begin{aligned} [p]x + [pb]y &= [pl] \\ [pb]x + [pb^2]y &= [pbl] \end{aligned}$$

whence eliminating x,

$$y\{[pb]^2 - [p][pb^2]\} = [pl][pb] - [p][pbl]$$

Now, if the first normal equation is divided by $[p]$, so that

$$x + \frac{[pb]}{[p]} y = \frac{[pl]}{[p]}$$

and from this equation each of the observation equations in succession is subtracted, there result the equations

$$\left(\frac{[pb]}{[p]} - b_1\right)y = \frac{[pl]}{[p]} - l_1 \quad \text{weight } p_1$$

$$\left(\frac{[pb]}{[p]} - b_2\right)y = \frac{[pl]}{[p]} - l_2 \quad \text{''} \quad p_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The normal equation for finding y from these equations is

$$\{[p][pb^2] - [pb]^2\}y = [p][pbl] - [pb][pl]$$

the same as results from the elimination of x in the normal equations.

This process is specially convenient if the original observation equations are numerous and the coefficients b_1, b_2, . . . and the terms l_1, l_2, . . . large and not widely different.

Ex. To solve the equations in Ex. 3, Art. 83.

The mean of the equations is

$$x - 0.97y = -1.97$$

Subtract each equation from this mean equation, and

$$\begin{aligned} +0.62y &= +0.01 \\ +0.47y &= -0.07 \\ +0.26y &= +0.03 \\ -0.01y &= -0.02 \\ -0.25y &= +0.03 \\ -1.08y &= 0.00 \end{aligned}$$

The normal equation formed from these equations is

$$+1.91y = -0.27$$

$$\text{and } \therefore \qquad y = -0^s.014$$

Substitute for y its value in the mean equation, and

$$x = -1^s.95$$

105. Again, we may take advantage of the presence in the problem of some arbitrary quantity to which a convenient value may be assigned. Thus, to find the difference of the coefficients of expansion of two standards A and B

from observed differences of length at certain fixed temperatures.

Let x = the excess in length of A over B at an arbitrary temperature t_0,

y = the excess of the coefficient of expansion of A over that of B,

$l_1, l_2, \ldots$ = the observed differences in length at temperatures $t_1, t_2, \ldots$ and whose weights are $p_1, p_2, \ldots$

We have then the observation equations

$$\begin{array}{ll} x + (t_1 - t_0)y - l_1 = v_1 & \text{weight } p_1 \\ x + (t_2 - t_0)y - l_2 = v_2 & \text{weight } p_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \end{array} \tag{1}$$

and the normal equations

$$\begin{array}{ll} [p]x + \{[pt] - [p]t_0\}y = [pl] & \\ \{[pt] - [p]t_0\}x + [p(t - t_0)^2]\,y & = [(t - t_0)pl] \end{array} \tag{2}$$

As the value of t_0 is arbitrary, the normal equations will be simplified by taking it equal to the weighted mean of the temperatures; that is,

$$t_0 = \frac{[pt]}{[p]} \tag{3}$$

and they will then become

$$\begin{array}{ll} [p]x & = [pl] \\ \qquad [p(t - t_0)^2]y & = [(t - t_0)pl] \end{array} \tag{4}$$

from which x and y are found at once, with their weights at the mean temperature t_0.

If the values of l are numerically large it will lessen the labor of finding the value of y if the mean value of x found from

$$[p]x = [pl]$$

is substituted in the observation equations before the normal equation in y is formed. We should then have

$$[p(t - t_0)^2]y = [p(t - t_0)(l - x)]$$

from which to find y.

It is evident that the value of y found in this way is the same as before. For

$$[p(t - t_0)(l - x)] = [p(t - t_0)l] - x\{[pt] - [p]t_0\}$$
$$= [p(t - t_0)l]$$

since the coefficient of x is equal to zero from Eq. 3.

The quantity $l - x$ comes in very conveniently in computing the residuals v in finding the precision.

The precision.—If n is the number of observations, the number of unknowns being 2, we have for the m. s. e. μ of a single observation

$$\mu = \sqrt{\frac{[pvv]}{n - 2}}$$

and

$$\mu_x = \frac{\mu}{\sqrt{[p]}}$$

$$\mu_y = \frac{\mu}{\sqrt{[p(t - t_0)^2]}}$$

The length at any temperature t' is

$$x + (t' - t_0)y$$

and its m. s. e. $\mu_{x'}$ is found from

$$\mu_{x'}^2 = \mu_x^2 + (t' - t_0)^2\mu_y^2$$
$$= \frac{\mu^2}{[p]} + \frac{(t' - t_0)^2}{[p(t - t_0)^2]}\mu^2$$

The weight is greatest when $\mu_{x'}$ is least, that is, when

$$t' = t_0 = \frac{[pt]}{[p]}$$

Ex. The following were among the observations made for the determination of the difference of length between the Lake Survey Standard Bar and Yard; and also for the difference between their coefficients of expansion. The unit is $\frac{1}{100000}$ inch.

Required the difference of length at 62° Fahr. and at any other temperature t.

Date.	Observed temp. (t).	Bar — Yard (l).	Weight (p).
1872, March 5	24°.7	791	5
" 14	37°.1	811	1
" 26	61°.7	833	6
April 4	49°.3	820	6
" 12	66°.8	847	8
" 20	71°.5	849	8

Let $x =$ the most probable difference between Bar and Yard at 62° Fahr.,
$y =$ the most probable difference between coefficients of expansion of Bar and Yard.

The observation equations will be of the form

$$x + (t - 62)y - l = v$$

The computation is arranged in tabular form as follows:

p	pt	pl	$t - t_0$	$l - x$
5	123.5	3955	− 32.2	− 40
1	37.1	811	− 19.8	− 20
6	370.2	4998	+ 4.8	+ 2
6	295.8	4920	− 7.6	− 11
8	534.4	6776	+ 9.9	+ 16
8	572.0	6792	+ 14.6	+ 18
34	1933.0	28252		
	$t_0 = 56°.9$	$x = 831$		

$p(t - t_0)^2$	$p(t - t_0)(l - x)$	$y(t - t_0)$	v	pvv
5184.20	6440.0	− 40.6	− 0.6	1.8
392.04	396.0	− 24.9	− 4.9	24.0
138.24	57.6	+ 6.0	+ 4.0	96.0
346.56	501.6	− 9.6	+ 1.4	11.8
784.08	1267.2	+ 12.5	− 3.5	98.0
1705.28	2102.4	+ 18.4	+ 0.4	1.3
8550.40	10764.8			232.9
$y = 1.26$				

$$\mu = \sqrt{\frac{232.9}{6-2}} = 7.6$$

$$\mu_x = \frac{7.6}{\sqrt{34}} = 1.3$$

$$\mu_y = \frac{7.6}{\sqrt{8550.40}} = 0.1$$

$$\begin{array}{lrr} \text{Value of } x = & 831 & \text{at } 56^\circ.9 \\ & 6.4 & 5^\circ.1 \\ \hline = & 837.4 & 62^\circ.0 \end{array}$$

$$\mu_{62}{}^2 = (1.3)^2 + (5.1)^2 \times (0.1)^2 = 1.9$$

These values may be checked by computing by the ordinary process.

The application of the preceding method to the solution of this problem is due to Mr. E. S. Wheeler.

106. It often happens that of two series of observed quantities one is a function of the other, and that, therefore, the observed quantities may be regarded as co-ordinates of points in the curve which represents the relation connecting them. From the plot of the points the general form of the curve can be judged, and then the special form which satisfies the observations may be found as follows.

Let us investigate the important practical case where the co-ordinates have been equally well measured and the curve is a straight line, both co-ordinates being regarded fallible.

Let $x_1, y_1 : x_2, y_2 : \ldots\ x_n, y_n$ be the observed values of the co-ordinates, and $X_1, Y_1 : X_2, Y_2 : \ldots\ X_n, Y_n$ their most probable values.

The conditions to be satisfied are:

$$\begin{aligned} Y_1 &= aX_1 + b \\ Y_2 &= aX_2 + b \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ Y_n &= aX_n + b \end{aligned} \qquad (1)$$

where a and b are constants, to be found from the observations. Also

$$(X_1 - x_1)^2 + (Y_1 - y_1)^2 + \ldots = \text{a min.} \qquad (2)$$

Eliminate the Y's by substituting from equations 1 in (2), and then

$$(X_1 - x_1)^2 + (aX_1 + b - y_1)^2 + \ldots = \text{a minimum.}$$

Differentiate this equation with respect to the independent variables $X_1, X_2, \ldots X_n$, and

$$\begin{aligned} X_1 - x_1 + a(aX_1 + b - y_1) &= 0 \\ X_2 - x_2 + a(aX_2 + b - y_2) &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \\ X_n - x_n + a(aX_n + b - y_n) &= 0 \end{aligned}$$

from which equations $X_1, X_2, \ldots X_n$ may be expressed in terms of a, b and the known quantities $x_1, y_1; \ldots; x_n, y_n$.

Again, differentiate with respect to a, b, and we find

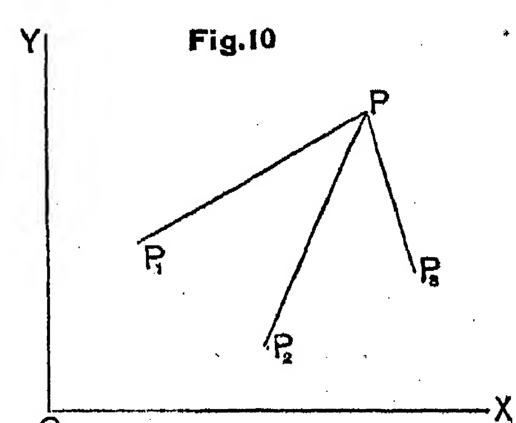
$$\begin{aligned} a[X^2] + b[X] &= [Xy] \\ a[X] + bn \quad &= [y] \end{aligned}$$

Hence a and b are known, and therefore the equation of the straight line is known.

The same mode of reasoning may be applied to the passing of a curve of a specified form through n points, whether their co-ordinates have been equally well measured or not.

The problem of passing a line which should deviate as little as possible from the positions of several given points was discussed by Lambert* as long ago as 1765, thirty years before Gauss first made use of the method of least squares. Lambert reasoned that if the line is supposed drawn the points should deviate as much on the one side as on the other, and hence that the line would pass through the centre of gravity of the points.

107. **Pothenot's Problem.**—In topographical work the three-point problem, usually called Pothenot's problem, is of great importance. It may be stated as follows:



Let P_1, P_2, P_3 be three points whose positions are known—as, for example, by their co-ordinates with reference to known axes—and P a point at which the angles $P_1PP_2 = \varphi_1 : P_2PP_3 = \varphi_2$ are

* *Beyträge zum Gebrauche der Mathematik.* Berlin, 1765.

observed. It is required from these data to determine the position of P.

The geometrical solution is simple. Plot the points P_1, P_2, P_3. Describe on the line P_1P_2 a circle containing an angle equal to φ_1, and on P_2P_3 a circle containing an angle equal to φ_2. The intersection of these circles will be the point required.

A solution much used in practice—as, for example, in plotting soundings—is to lay off on a piece of tracing-linen two adjacent angles equal to φ_1, φ_2, and then move this figure over the map on which the points P_1, P_2, P_3 are plotted until the directions PP_1, PP_2, PP_3 lie over the points P_1, P_2, P_3 respectively. The position of P is then pricked through.

So far as the plotting of soundings is concerned, the above is in general sufficient; but in the location of important secondary points with reference to known primary points greater precision is necessary.

The position of P may be computed trigonometrically as follows: Since the positions of the points P_1, P_2, P_3 are known, the lengths of the lines P_1P_2, P_2P_3, P_1P_3 are known, and, therefore, the angle $P_1P_2P_3$ can be found. Call its value β.

Denote the angles PP_1P_2, PP_3P_2 by α_1, α_2 respectively.

The sum of the angles of each of the triangles PP_2P_3, PP_1P_2 being 180°, we have

$$\beta + \alpha_1 + \varphi_1 + \alpha_2 + \varphi_2 = 360°$$

and, therefore,

$$\alpha_1 + \alpha_2 = 360° - \varphi_1 - \varphi_2 - \beta$$

a known quantity.

Again,

$$\frac{PP_2}{\sin \alpha_1} = \frac{P_1P_2}{\sin \varphi_1} \qquad \frac{PP_2}{\sin \alpha_2} = \frac{P_2P_3}{\sin \varphi_2}$$

$$\therefore \frac{\sin \alpha_2}{\sin \alpha_1} = \frac{P_1P_2 \sin \varphi_2}{P_2P_3 \sin \varphi_1} = \tan \gamma \text{ suppose,}$$

and

$$\frac{\sin \alpha_1 + \sin \alpha_2}{\sin \alpha_1 - \sin \alpha_2} = \frac{1 + \tan \gamma}{1 - \tan \gamma}.$$

or

$$\cot \tfrac{1}{2}(\alpha_1 - \alpha_2) = \tan(\gamma + 45) \cot \tfrac{1}{2}(\alpha_1 + \alpha_2)$$

and, therefore, $\alpha_1 - \alpha_2$ is known. Combining with the value of $\alpha_1 + \alpha_2$, we find α_1 and α_2.

Hence the position of P is completely determined.

It often happens, as in the determination of the position of a light-house, for example, that more than three points are sighted at from the point occupied, and that there are more observations than are necessary to locate the point. Its most probable position can consequently be found. We proceed to show the method of finding it.

Let $P_1, P_2, \ldots$ be the known points in order of azimuth, as seen from the unknown point P. $X_1, Y_1 : X_2, Y_2; \ldots$ the co-ordinates of $P_1, P_2, \ldots$ referred to some known system of rectangular axes, preferably the parallel and meridian at the point chosen as origin of co-ordinates. $\theta_1, \theta_2, \ldots$ the angles which the most probable positions of $PP_1, PP_2, \ldots$ make with the axis of x. $\theta_1', \theta_2', \ldots$ approximate values of these angles as computed from the co-ordinates, and $\vartheta_1, \vartheta_2, \ldots$ corrections to these approximate values, so that

$$\theta_1 = \theta_1' + \vartheta_1,\ \theta_2 = \theta_2' + \vartheta_2, \ldots$$

$\varphi_1, \varphi_2, \ldots$ the angles observed at P between the directions $PP_2, PP_3, \ldots$ and the initial direction PP_1.

$X, Y, \ldots$ the co-ordinates of P referred to the same axes as $X_1, Y_1 : X_2, Y_2; \ldots$ Suppose X', Y' to be close approximations to the values of X, Y, found graphically, and x, y corrections to these values, so that

$$X = X' + x,\ Y = Y' + y,$$

Now,

$$\tan \theta_1 = \frac{Y - Y_1}{X - X_1}$$

or

$$\tan(\theta_1' + \vartheta_1) = \frac{Y' - Y_1 + y}{X' - X_1 + x}$$

Take logs. of both members and expand, then

$$\log \tan \theta_1' + \delta_1 \vartheta_1 = \log (Y' - Y_1) + \delta_2 y - \log (X' - X_1) - \delta_3 x$$

where δ_1, δ_2, δ_3 are tabular differences, as explained in Art. 7.

But

$$\tan \theta_1' = \frac{Y' - Y_1}{X' - X_1}, \text{ and is, therefore, known.}$$

Hence

$$\vartheta_1 = \frac{\delta_2}{\delta_1} y - \frac{\delta_3}{\delta_1} x$$

$$= a_1 x + b_1 y \text{ suppose,}$$

or

$$\theta_1 = \theta_1' + a_1 x + b_1 y$$

Similarly

$$\theta_2 = \theta_2' + a_2 x + b_2 y$$

Now, comparing computed and observed values of the angle P_1PP_2,

$$\begin{aligned} -\varphi_1 &= \theta_1 - \theta_2 \\ &= \theta_1' - \theta_2' + (a_1 - a_2)x + (b_1 - b_2)y \end{aligned}$$

Calling the quantity

$$-\varphi_1 - (\theta_1' - \theta_2') = l_1$$

the observation equation may be written

$$(a_1 - a_2)x + (b_1 - b_2)y = l_1$$

An observation equation for each of the other angles φ_2, φ_3, . . . observed at P may be formed in the same way.

In the expansions above we have used log. differences as being most convenient and as leading to results close enough in problems of this kind. Five-place tables are sufficient; indeed, in most cases four-place are ample.

On this problem consult Bessel, Zach's *Monatliche Correspondenz*, vol. xxvii. p. 222; Gerling, *Pothenotsche Aufgabe*, Marburg, 1840; Gauss, *Astron. Nachr.*, vol. i. No. 6; Schott, *C. S. Report*, 1864, App. 13; Petzold, *Zeitschr. für Vermess.*, vol. xii. p. 227.

Ex. Given the rectangular co-ordinates of six known points, and the angles observed at the point P whose position is to be determined, as follows (*C. S. Report*, 1864, App. 13):

$$\begin{array}{lll} X_1 = 1845.0 & Y_1 = 5534.0 & \varphi_1 = 61^\circ\ 12'\ 10'' \\ X_2 = 1485.0 & Y_2 = 2486.7 & \varphi_2 = 97^\circ\ 48'\ 27'' \\ X_3 = 0.0 & Y_3 = 0.0 & \varphi_3 = 195^\circ\ 55'\ 56'' \\ X_4 = 4418.2 & Y_4 = 1416.7 & \varphi_4 = 205^\circ\ 35'\ 04'' \\ X_5 = 6163.8 & Y_5 = 398.1 & \varphi_5 = 215^\circ\ 53'\ 44'' \\ X_6 = 5810.6 & Y_6 = 1255.7 & \end{array}$$

to find the co-ordinates of P.

By plotting the points we may take off the approximate values

$$X' = 3448 \qquad Y' = 2440$$

Following the form laid down in the preceding Art., and using five-place log. tables (Art. 7),

$$\begin{aligned} \log\,(3094 - y) &= 3.49052 \quad - 14.0y \\ \log\,(-\,1603 - x) &= 3.20493n + 27.1x \end{aligned}$$

$$\therefore\ \log\tan\Theta_1' + \delta_1\vartheta_1 = 0.28559n - 27.1x - 14.0y$$

Now, $\qquad \Theta_1' = 117^\circ\ 23'\ 19''$ and $\delta_1 = -31$ for $1'$

Hence

$$\Theta_1 = 117^\circ\ 23'\ 19'' + 52.5x + 27.1y$$

Similarly

$$\begin{aligned} \Theta_2 &= 178^\circ\ 38'\ 14'' + 2.5x + 103.6y \\ \Theta_3 &= 215^\circ\ 17'\ 07'' - 28.0x + 39.6y \\ \Theta_4 &= 313^\circ\ 28'\ 28'' - 103.2x - 97.8y \\ \Theta_5 &= 323^\circ\ 03'\ 45'' - 35.6x - 47.3y \\ \Theta_6 &= 333^\circ\ 22'\ 37'' - 35.6x - 71.0y \end{aligned}$$

Subtracting Θ_1 from each of the values Θ_2, Θ_3, . . . in succession, and comparing with the measured values φ_1, φ_2, . . ., we have the observation equations

$$\begin{aligned} 50.0x - 76.5y &= 165 \\ 80.5x - 12.5y &= 321 \\ 155.7x + 124.9y &= 553 \\ 88.1x + 74.4y &= 322 \\ 88.1x + 98.1y &= 334 \end{aligned}$$

Hence the normal equations

$$48746x + 29813y = 177986$$
$$29813x + 36767y = 109157$$

The solution gives

$$x = +3.66 \quad \text{weight } 24572$$
$$y = -0.02 \quad \text{weight } 18533$$

which added to the approximate values above give the final values

$$X = 3451.66 \qquad Y = 2439.98$$

Substitute for x, y in the observation equations, and we have the residuals

$$v_1 = +19.5,\ v_2 = -27.8,\ v_3 = +8.1,\ v_4 = +1.0,\ v_5 = -13.5$$

Hence, since $[vv] = 1402$,

$$\mu = \sqrt{\frac{1402}{5-2}} = 22$$

and

$$\mu_X = \frac{22}{\sqrt{24572}} = 0.14$$

$$\mu_Y = \frac{22}{\sqrt{18533}} = 0.16$$

CHAPTER V.

ADJUSTMENT OF CONDITION OBSERVATIONS.

108. We now take up the third division of the subject as laid down in Art. 39. So far the quantities we have dealt with, whether directly observed or functions of the quantities observed, have been independent of one another; but if they are not independent of one another—that is, if they must satisfy exactly certain relations that exist *à priori* and are entirely separate from any relations demanded by observation—they are said to be *conditioned* by these relations.

All problems relating to condition observations may be solved by the rules laid down in the preceding chapters.

Let, with the usual notation, $V_1, V_2, \ldots V_n$ denote the most probable values of n directly observed quantities $M_1, M_2, \ldots M_n$ whose weights are $p_1, p_2, \ldots p_n$ respectively. Let the n_c conditions to be satisfied exactly by the most probable values, when expressed by equations reduced to the linear form, be

$$\begin{array}{l} a'V_1 + a''V_2 + \ldots - L' = 0 \\ b'V_1 + b''V_2 + \ldots - L'' = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (1)$$

where $a', a'', \ldots; b', b'', \ldots; \ldots; L', L'', \ldots$ are known constants.

If $v_1, v_2, \ldots v_n$ denote the most probable corrections to the observed values, so that

$$\begin{array}{l} V_1 - M_1 = v_1 \\ V_2 - M_2 = v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ V_n - M_n = v_n \end{array} \qquad (2)$$

we have the reduced condition equations

$$a'v_1 + a''v_2 + \ldots - l' = 0$$
$$b'v_1 + b''v_2 + \ldots - l'' = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

or

$$[av] - l' = 0$$
$$[bv] - l'' = 0 \qquad (3)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where $l' = L' - [aM]$, $l'' = L'' - [bM]$, . . ., and are, therefore, known quantities.

The most probable system of corrections is that which makes

$$[pvv] = \text{a minimum, } \omega \text{ suppose.}$$

The problem is to solve this minimum function when the corrections v are subject to the above n_c conditions.

Direct Solution—Method of Independent Unknowns.

109. It is plain that n_c of the corrections can, by means of the condition equations, be expressed in terms of the remaining $n - n_c$ corrections, and that by substituting these n_c values in the minimum function we should have a reduced minimum function containing $n - n_c$ independent unknowns. This function can be found in the usual way by equating to zero its differential coefficients with respect to each unknown in succession. The $n - n_c$ resulting equations, taken in connection with the n_c condition equations, determine the n corrections $v_1, v_2, \ldots v_n$. Thence $[pvv]$ is found.

The solution of the $n - n_c$ equations can be carried through by any of the methods of Chapter IV. The precision of the adjusted values, or of any function of them, can also be found as in Chapter IV.

Ex. 1. Take that already solved in Ex. 4, Art. 83.

Let v_1, v_2, v_3, v_4, v_5 be the most probable corrections to the measured angles, then the conditions to be satisfied are

$$PSB + v_4 = FSB + v_3 - FSP - v_1$$
$$OSB + v_5 = FSB + v_3 - FSO - v_2$$

Substituting for PSB, FSB, etc., their measured values, the condition equations may be written

$$v_1 - v_3 + v_4 = -0.76$$
$$v_2 - v_3 + v_5 = -1.66$$

with

$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7v_4^2 + 4v_5^2 = \text{a min.}$$

Substitute for v_4, v_5 in the minimum equation, and

$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7(v_1 - v_3 + 0.76)^2 + 4(v_2 - v_3 + 1.66)^2 = \text{a min.}$$

Hence, differentiating with respect to v_1, v_2, v_3 as independent variables, we have the normal equations

$$\begin{aligned} 12v_1 \qquad\quad - 7v_3 &= -\ \ 5.32 \\ 11v_2 - 4v_3 &= -\ \ 6.64 \\ -7v_1 - 4v_2 + 15v_3 &= \quad 11.96 \end{aligned}$$

whence

$$v_1 = -0''.05, \quad v_2 = -0''.36, \quad v_3 = +0''.68$$

and from the condition equations

$$v_4 = -0''.03 \quad v_5 = -0''.62$$

These results are the same as those already found on p. 149.

Ex. 2. The angles A, B, C of a spherical triangle are equally well measured; required the adjusted values and their weights.

The condition equation to be satisfied is

$$A + B + C = 180 + \varepsilon \tag{1}$$

where ε is the spherical excess of the triangle.

Putting $M_1 + v_1$, $M_2 + v_2$, $M_3 + v_3$ for A, B, C, the condition equation becomes

$$\begin{aligned} v_1 + v_2 + v_3 &= 180 + \varepsilon - [M] \\ &= l \text{ suppose} \end{aligned} \tag{2}$$

Also

$$v_1^2 + v_2^2 + v_3^2 = \text{a min.}$$

Substitute for v_3 from (2) in the minimum function, and

$$v_1^2 + v_2^2 + (v_1 + v_2 - l)^2 = \text{a min.}$$

Differentiating with respect to the independent variables v_1, v_2, and

$$\begin{aligned} 2v_1 + v_2 &= l \\ v_1 + 2v_2 &= l \end{aligned} \qquad (3)$$

which give

$$v_1 = v_2 = \frac{l}{3}$$

Also from Eq. 2,

$$v_3 = \frac{l}{3}$$

Hence *the correction to each angle is one-third of the difference of the theoretical and measured sums of the three angles.*

To find the weight of the adjusted value of an angle, as A.

The function is

$$dF = v_1$$

Hence, following the method of Art. 101 (b),

$$u_F = \frac{1}{\text{wt.}} = [GQ]$$

where $G_1 = 1$, and Q_1, Q_2 are found from

$$\begin{aligned} 2Q_1 + Q_2 &= 1 \\ Q_1 + 2Q_2 &= 0 \end{aligned}$$

that is, weight of $A = \frac{3}{2}$ if weight of measured value is unity.

Check. Weight of direct measure of A $= 1$

Wt. of indirect meas. $(= 180 + \varepsilon - B - C)$ of $A = \frac{1}{2}$

Weight of mean $= 1\frac{1}{2}$

as already found.

Ex. 3. To find the weight of a side, a, in a triangle all of whose angles have been equally well measured, the base, b, being free from error.

Here
$$F = a = b\,\frac{\sin A}{\sin B}$$

$$\therefore dF = a \sin 1'' (\cot A\, v_1 - \cot B\, v_2)$$

The weight is found from

$$u_F = a \sin 1'' \cot A\; Q_1 - a \sin 1'' \cot B\; Q_2$$

where Q_1, Q_2 satisfy the equations (Art. 101)

$$\begin{aligned} 2Q_1 + Q_2 &= a \sin 1'' \cot A \\ Q_1 + 2Q_2 &= -a \sin 1'' \cot B \end{aligned}$$

Hence

$$u_F = \tfrac{2}{3} a^2 \sin^2 1'' (\cot^2 A + \cot^2 B + \cot A \cot B)$$

Ex. 4. The measured values of the angles of a triangle have the same weight. Show that if the corrections to the angles are expressed in terms of the corrections to the log. sines of the angles, and the corrections to these log. sines found by treating them as observed quantities, the same results will be obtained as in Ex. 2.

For example take 50°, 60°, 70° 00′ 30″.

Indirect Solution—Method of Correlates.

110. If the unknowns in the condition equations are much entangled the direct solution would be very laborious. It is in general, therefore, advisable, instead of eliminating the n_c unknowns directly, to do so indirectly by means of undetermined multipliers, or correlates, as they are called.

If we multiply the condition equations 3, Art. 108, in order by the correlates k', k'', . . ., we may write

$$\omega = k'([av] - l') + k''([bv] - l'') + \ldots + [pvv] = \text{a min.} \quad (1)$$

and determine k', k'', . . . accordingly.

By differentiation,

$$d\omega = (a'k' + b'k'' + \ldots + 2p_1v_1)dv_1 \\ + (a''k' + b''k'' + \ldots + 2p_2v_2)dv_2 + \ldots \quad (2)$$

If we place equal to zero the coefficients of n_c of the differentials dv_1, dv_2, . . . we shall have n_c equations from which to find k', k'', . . . Substitute these n_c values in the expression for $d\omega$, and there will remain $n - n_c$ differentials which are independent of one another. In order that the function may satisfy the condition of a minimum, the coefficients of each of these differentials must be equal to zero. This gives $n - n_c$ equations, which equations, taken in connection with the n_c condition equations, give the n unknowns v_1, v_2, . . . v_n.

The practical solution would, therefore, be: Form n equations by placing equal to zero the differential coefficients of the minimum function with respect to each of the quantities v_1, v_2, . . . v_n. From these n equations and the n_c condition equations determine the $n + n_c$ unknowns k', k'', . . ., v_1, v_2, . . ., and thence the function $[pvv]$.

In carrying this out the form of the differential equation 2 shows that it would be advantageous to multiply the minimum equation by $-\frac{1}{2}$, and so write (1) in the form

$$k'([av]-l')+k''([bv]-l'')+\ldots-\tfrac{1}{2}[pvv]=\text{a min.} \qquad (3)$$

Differentiating, we have the n correlate equations

$$\begin{array}{l} a'k'+b'k''+\ldots=p_1v_1 \\ a''k'+b''k''+\ldots=p_2v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (4)$$

Substituting for $v_1, v_2, \ldots$ in the condition equations their values derived from these equations, and the normal equations result. They are

$$\begin{array}{l} \left[\frac{aa}{p}\right]k'+\left[\frac{ab}{p}\right]k''+\ldots=l' \\ \left[\frac{ab}{p}\right]k'+\left[\frac{bb}{p}\right]k''+\ldots=l'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (5)$$

Solving, we obtain $k', k'', \ldots$, and thence $v_1, v_2, \ldots$ from (4), and $V_1, V_2, \ldots$ from (2), Art. 108.

The normal equations may be written

$$\begin{array}{l} [uaa]k'+[uab]k''+\ldots=l' \\ [uab]k+[ubb]k''+\ldots=l'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \qquad (6)$$

where $u_1, u_2, \ldots$ denote the reciprocals of the weights $p_1, p_2, \ldots$. The form of these equations shows that the coefficients $[uaa]$, $[uab]$, ... may be computed as in Art. 85, the corresponding scheme being

		k'	k''	k'''	...
v_1	u_1	a'	b'	c'	...
v_2	u_2	a''	b''	c''	...
.	.	.	.	.	...

If the elimination of the normal equations is performed by the method of substitution (Art 86), we have, by collecting the first equations of the successive groups,

$$\begin{aligned}
[uaa]k' + [uab]k'' \ &+ [uac]k''' + \ldots = l' \\
+ [ubb.1]k'' &+ [ubc.1]k''' + \ldots = l''.1 \\
&+ [ucc.2]k''' + \ldots = l'''.2 \\
&\quad . \quad . \quad . \quad . \quad .
\end{aligned} \tag{7}$$

where l', $l''.1$, $l'''.2$, . . . correspond to $[al]$, $[bl.1]$, $[cl.2]$, . . . respectively.

These equations being precisely similar in form to Eq. 8, Art. 86, the elimination gives (see Art. 96)

$$\begin{aligned}
k' &= \frac{l'}{[uaa]} + \frac{l''.1}{[ubb.1]}R' + \frac{l'''.2}{[ucc.2]}R'' + \ldots \\
k'' &= \qquad\quad \frac{l''.1}{[ubb.1]} \quad + \frac{l'''.2}{[ucc.2]}S'' + \ldots \\
&\qquad . \quad . \quad . \quad .
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
0 &= \frac{[uab]}{[uaa]} + R' \\
0 &= \frac{[uac]}{[uaa]} + \frac{[ubc.1]}{[ubb.1]}R' + R'' \\
&\quad . \quad . \quad . \quad . \quad . \\
0 &= \frac{[ubc.1]}{[ubb.1]} + S'' \\
&\quad . \quad . \quad . \quad .
\end{aligned} \tag{9}$$

$$\begin{aligned}
l''.1 \ &= l'R' + l'' \\
l'''.2 &= l'R'' + l''S'' + l''' \\
&\quad . \quad . \quad . \quad . \quad .
\end{aligned} \tag{10}$$

Ex. 1. Take that solved in Ex. 1, Art. 109.
The condition equations are

$$\begin{aligned} v_1 - v_3 + v_4 &= -0.76 \\ v_2 - v_3 + v_5 &= -1.66 \end{aligned} \tag{1}$$

The correlate equations consequently are

$$\begin{aligned} k' \qquad &= 5v_1 \\ k'' &= 7v_2 \\ -k' - k'' &= 4v_3 \\ k' \qquad &= 7v_4 \\ k'' &= 4v_5 \end{aligned} \tag{2}$$

To form the normal equations we may substitute for $v_1, v_2, \ldots$ from (2) in (1), or proceed by means of the tabular form on p. 218. We find

$$\begin{aligned} 0.5929k' + 0.25k'' &= -0.76 \\ 0.25k' + 0.6429k'' &= -1.66 \end{aligned}$$

The solution of these equations gives

$$k' = -0.230 \qquad k'' = -2.492$$

whence, from the correlate equations,

$$v_1 = -0''.05,\ v_2 = -0''.36,\ v_3 = +0''.68,\ v_4 = -0''.03,\ v_5 = -0''.62$$

Check. The results satisfy the condition equations.

Ex. 2. The angles, A, B, C, of a spherical triangle are measured with their weights, p_1, p_2, p_3; required their adjusted values.

The condition equation may be written (see Ex. 2, Art. 109)

$$v_1 + v_2 + v_3 = l$$

with

$$[pv^2] = \text{a min.}$$

The correlate equations are

$$\begin{aligned} k &= p_1 v_1 \\ k &= p_2 v_2 \\ k &= p_3 v_3 \end{aligned}$$

and the normal equation

$$[u]k = l$$

$$\therefore\ v_1 = \frac{u_1}{[u]}l, \quad v_2 = \frac{u_2}{[u]}l, \quad v_3 = \frac{u_3}{[u]}l$$

Hence the adjusted values are known.

When the weights are equal, then

$$v_1 = v_2 = v_3 = \tfrac{1}{3}l$$

the same results as in Ex. 2, Art. 109.

Note. If a condition equation is of the form

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = l$$

the weights of the measured values being $p_1, p_2, \ldots p_n$, then, proceeding as in the above, we have

$$v_1 = \frac{u_1a_1}{[uaa]}\,l,\ v_2 = \frac{u_2a_2}{[uaa]}\,l,\ \ldots$$

This result is very important and will be often referred to.

Ex. 3. At U. S. Coast Survey station Pine Mt. the following were the angles observed between the surrounding stations in order of azimuth:

Jocelyne–Deepwater,	65°	11′	52″.500	weight 3
Deepwater–Deakyne,	66°	24′	15″.553	" 3
Deakyne–Burden,	87°	02′	24″.703	" 3
Burden–Jocelyne,	141°	21′	21″.757	" 1

required their most probable values.

The condition to be satisfied is that the sum of the angles should be 360°. Now,

$$\begin{aligned} \text{sum of measured values} &= 359^\circ\ 59'\ 54''.513 \\ \text{theoretical sum} &= 360^\circ\ 00'\ 00''.000 \\ \therefore \text{residual error} &= 5''.487 \end{aligned}$$

Hence, as in the preceding example,

$$\text{correction to each of first three angles} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+1} \times 5''.487$$

$$= 0''.914$$

$$\text{correction to fourth angle} = \frac{1}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+1} \times 5''.487$$

$$= 2''.744$$

Ex. 4. Given in a triangle, of which the longest side is 4 miles, the three angles measured with equal care and with the values 1″, 5″, 179° 59′ 58″; adjust the triangle.

[Angles as small as single seconds occur in practice. In the primary triangulation (1877) of the Great Lakes, carried out by the U. S. Engineers, in the neighborhood of the Chicago base two angles were measured with values 0″.815 and 1″.185 respectively.]

Ex. 5. "In order to find the content of a piece of ground, I measured with a common circumferentor and chain the bearings and lengths of its several sides. But upon casting up the difference of latitude and departure

I discovered that some error had been contracted in taking the dimensions. Now, it is required to compute the area of this enclosure on the *most probable supposition* of this error.

"Let *ABCDE* be a survey accurately protracted according to the measured lengths and bearings of the sides *AB*, *BC*, . . . *A* the place of beginning, *E* of ending, *AG* a meridian, *AF*, *FE* the errors in latitude and departure. Now, the problem requires us to make such changes in the positions of the points *B*, *C*, . . . that we may remove the errors *AF*, *FE*—in other words, that *E* may coincide with *A*; and these changes must be made in the most probable manner. We have, therefore, to fulfil the three following conditions:

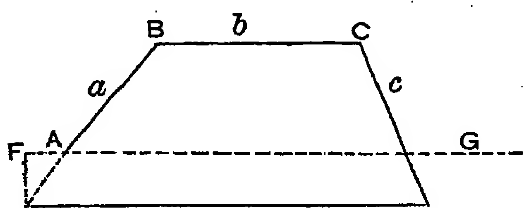


Fig. 11

"All the changes in departure must remove the error in departure *EF*.

"All the changes in latitude must remove the error in latitude *AF*.

"The probability of these changes must be a maximum."

Let $a_1, a_2, a_3, \ldots$; $\vartheta_1, \vartheta_2, \vartheta_3, \ldots$; denote the measured lengths and bearings of the sides *AB*, *BC*, . . ., and $x_1, x_2, x_3, \ldots$; $y_1, y_2, y_3, \ldots$ their most probable corrections.

Now, since the corrected latitudes must balance, and the corrected departures must also balance, we have the conditions

$$(a_1 + x_1) \cos (\vartheta_1 + y_1) + (a_2 + x_2) \cos (\vartheta_2 + y_2) + \ldots = 0$$
$$(a_1 + x_1) \sin (\vartheta_1 + y_1) + (a_2 + x_2) \sin (\vartheta_2 + y_2) + \ldots = 0$$

or, reducing to the linear form,

$$\left.\begin{aligned} \cos\vartheta_1\, x_1 - a_1 \sin\vartheta_1\, y_1 + \cos\vartheta_2\, x_2 - a_2 \sin\vartheta_2\, y_2 + \ldots + [a \cos \vartheta] &= 0 \\ \sin\vartheta_1\, x_1 + a_1 \cos\vartheta_1\, y_1 + \sin\vartheta_2\, x_2 + a_2 \cos\vartheta_2\, y_2 + \ldots + [a \sin \vartheta] &= 0 \end{aligned}\right\} \quad (1)$$

with

$$[px^2] + [qy^2] = \text{a minimum},$$

the weights of $x_1, x_2, \ldots$; $y_1, y_2, \ldots$ being $p_1, p_2, \ldots$; $q_1, q_2, \ldots$ respectively.

Hence the correlate equations

$$\begin{aligned} + \cos\vartheta_1\, k' + \sin\vartheta_1\, k'' &= p_1 x_1 \\ - a_1 \sin\vartheta_1\, k' + a_1 \cos\vartheta_1\, k'' &= q_1 y_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \end{aligned} \quad (2)$$

and the normal equations

$$\left\{\left[\frac{\cos^2 \vartheta}{p}\right] + \left[\frac{a^2 \sin^2 \vartheta}{q}\right]\right\} k' + \left\{\left[\frac{\sin \vartheta \cos \vartheta}{p}\right] - \left[\frac{a^2 \sin \vartheta \cos \vartheta}{q}\right]\right\} k'' = -[a \cos \vartheta]$$

$$\left\{\left[\frac{\sin \vartheta \cos \vartheta}{p}\right] - \left[\frac{a^2 \sin \vartheta \cos \vartheta}{q}\right]\right\} k' + \left\{\left[\frac{\sin^2 \vartheta}{p}\right] + \left[\frac{a^2 \cos^2 \vartheta}{q}\right]\right\} k'' = -[a \sin \vartheta]$$

Now if we assume

$$p_1 = \frac{1}{a_1},\ p_2 = \frac{1}{a_2},\ \ldots$$

$$q_1 = a_1,\ q_2 = a_2,\ \ldots$$

as "this seems best to agree with the imperfections of the common instruments used in surveying," the normal equations reduce to

$$\begin{aligned} k'[a] &= -[a \cos \vartheta] \\ k''[a] &= -[a \sin \vartheta] \end{aligned}$$

from which k', k'' are known.

The corrections $x_1, x_2, \ldots; y_1\ y_2 \ldots$, are known from (2).

The errors in latitude (see Eq. 1) now reduce to

$$\cos\vartheta_1\, x_1 - a_1 \sin\vartheta_1\, y_1 = -a_1 \frac{[a \cos \vartheta]}{[a]}$$

$$\cos\vartheta_2\, x_2 - a_2 \sin\vartheta_2\, y_2 = -a_2 \frac{[a \cos \vartheta]}{[a]}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and the errors in departure to

$$\sin\vartheta_1\, x_1 + a_1 \cos\vartheta_1\, y_1 = -a_1 \frac{[a \sin \vartheta]}{[a]}$$

$$\sin\vartheta_2\, x_2 + a_2 \sin\vartheta_2\, y_2 = -a_2 \frac{[a \sin \vartheta]}{[a]}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence Bowditch's *rule for balancing a survey:* "Say as the sum of all the distances is to each particular distance, so is the whole error in departure to the correction of the corresponding departure, each correction being so applied as to diminish the whole error in departure. Proceed in same way for the correction in latitude."

This problem was proposed as a prize question by Robert Patterson, of Philadelphia, in vol. i. No. 3 of the *Analyst or Mathematical Museum*, edited by Dr. Adrain, of Reading, Pa., and published in 1807. In vol. i. No. 4 are two solutions—one by Bowditch, to whom the prize was awarded, and the other by Dr. Adrain. Adrain's mode of solution is nearly the same as by the ordinary Gaussian method. He employs undetermined multipliers or correlates, exactly as Gauss subsequently did. To Adrain, therefore, is due not only the first derivation of the exponential law of error, but its first application to geodetic work. See Appendix I.

To Find the Precision of the Adjusted Values, or of any Function of them.

111. The method of proceeding is the same as in Art. 101.

The first step is to find μ, the m. s. e. of a single observation, and next the weight, p_F, of the function, whence the m. s. e. of the function is given by

$$\mu \sqrt{u_F}$$

u_F being the reciprocal of the weight.

(a) To find μ.

In Art. 99 it was shown that in a system of observation equations the m. s. e. μ of an observation of the unit of weight is found from

$$\mu = \sqrt{\frac{[pvv]}{n - n_i}}$$

where $[pvv]$ is the sum of the weighted squares of the residuals v, n is the number of observation equations, and n_i the number of independent unknowns.

Hence in a system of condition equations, n being the number of observed quantities and n_c the number of conditions, the number of independent unknowns is $n - n_c$, and

$$\mu = \sqrt{\frac{[pvv]}{n - (n - n_c)}}$$

$$\mu = \sqrt{\frac{[pvv]}{n_c}} \qquad (1)$$

Lüroth's formula (Art. 99) may be used as a check on the value of μ.

Checks of $[pvv]$.—When the number of residuals is large, in order to guard against mistakes $[pvv]$ should be computed in at least two different ways. The following check methods will be found useful:

(α) The correlate equations 4, Art. 110, may be written

$$\sqrt{p_1}\, v_1 = \sqrt{u_1}\, a' k' + \sqrt{u_1}\, b' k'' + \ldots$$
$$\sqrt{p_2}\, v_2 = \sqrt{u_2}\, a'' k' + \sqrt{u_2}\, b'' k'' + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Square and add, and

$$\begin{aligned}[pvv] = [uaa]k'k' &+ 2[uab]k'k'' + 2[uac]k'k''' + \ldots \\ &+ [ubb]k''k'' + 2[ubc]k''k''' + \ldots \\ &\qquad + [ucc]k'''k''' + \ldots \qquad (2) \\ &\qquad\qquad + \ldots \end{aligned}$$

$$= [kl] \text{ from (6), Art. 110.}$$

(β) $[pvv] = [kl]$

$$\begin{aligned} &= l'k' + l''k'' + \ldots \\ &= l'\left(\frac{l'}{[uaa]} + \frac{l''.1}{[ubb.1]}R' + \frac{l'''.2}{[ucc.2]}R'' + \ldots\right) \\ &\qquad + l''\left(\frac{l''.1}{[ubb.1]} + \frac{l'''.2}{[ucc.2]}S'' + \ldots\right) \\ &\qquad\qquad + \ldots \\ &= \frac{(l')^2}{[uaa]} + \frac{(l''.1)^2}{[ubb.1]} + \frac{(l'''.2)^2}{[ucc.2]} + \ldots \qquad (3) \end{aligned}$$

by addition attending to Eq. 10, Art. 110.

This expression is very readily computed from the solution of the correlate normal equations, as shown in Ex. 2 following. Compare the computation of $[vv]$ from the scheme in Art. 100.

The sum $[pvv]$ can in general be computed more rapidly by these methods than by the direct process of summing the weighted squares of the residuals.

Ex. 1. The three angles of a triangle are measured with the weights p_1, p_2, p_3; required the mean-square error of a single observation.

Using the values of v_1, v_2, v_3 found in Ex. 2, Art. 110, we have

$$[pvv] = \frac{u_1 l^2}{[u]^2} + \frac{u_2 l^2}{[u]^2} + \frac{u_3 l^2}{[u]^2}$$
$$= \frac{l^2}{[u]}$$

Hence

$$\mu = \frac{l}{\sqrt{[u]}} \text{ since } n_c = 1$$

Check (1). $[pvv] = [kl]$

$$= \frac{l}{[u]} l$$

as before.

Check (2). $[pvv] = \frac{(l)^2}{[u]}$ directly from Eq. 3, since $[uaa] = 1$.

Ex. 2. To find the m. s. e. of a single observation in Ex. 1, Art. 110. The first step is to find the value of $[pvv]$. Three methods are given:

(1)

p	v	pvv
5	− 0.05	.0125
7	− 0.36	.9072
4	+ 0.68	1.8496
7	− 0.03	.0063
4	− 0.62	1.5376
		4.3132 = $[pvv]$

(2)

k	l	kl
− 0.230	− 0.76	0.1748
− 2.493	− 1.66	4.1384
		4.3132 = $[pvv]$

(3) From the solution of the correlate normal equations:

k'	k''	
+ 0.5929	+ 0.2500	− 0.76 = l'
+ 0.2500	+ 0.6429	− 1.66 = l''
+ 1.	+ 0.4217	− 1.2818 = $\frac{l'}{[u.aa]}$
	+ 0.5375	− 1.3395 = $l''.1$
	+ 1.	− 2.492 = $\frac{l''.1}{[ubb.1]}$

$$\therefore \quad [pvv] = 0.76 \times 1.2819 + 1.3395 \times 2.492$$
$$= 4.3136$$

Hence, the number of conditions being two,

$$\mu = \sqrt{\frac{4.31}{2}} = 1''.47$$

(b) To find u_F.

Let the function whose weight is to be found be

$$F = f(V_1, V_2, \ldots V_n) \tag{4}$$

and let it be conditioned by the n_c equations

$$\begin{aligned} f_1(V_1, V_2, \ldots V_n) &= 0 \\ f_2(V_1, V_2, \ldots V_n) &= 0 \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \end{aligned} \tag{5}$$

Expressing F in terms of the observed values, $M_1, M_2, \ldots M_n$, which are independent of one another, and reducing to the linear form, we have

$$dF = \frac{\partial F}{\partial M_1} v_1 + \frac{\partial F}{\partial M_2} v_2 + \ldots \tag{6}$$

Hence as in Art. 101,

$$u_F = u_1 \left(\frac{\partial F}{\partial M_1}\right)^2 + u_2 \left(\frac{\partial F}{\partial M_2}\right)^2 + \ldots \tag{7}$$

where $u_1, u_2, \ldots$ are the reciprocals of the weights of the observed values.

As it usually requires a long elimination to express F in terms of $M_1, M_2, \ldots M_n$ directly, it is better to compute $\frac{\partial F}{\partial M_1}, \frac{\partial F}{\partial M_2}, \ldots$ from the forms

$$\begin{aligned} \frac{\partial F}{\partial M_1} &= \frac{\partial F}{\partial V_1} \frac{\partial V_1}{\partial M_1} + \frac{\partial F}{\partial V_2} \frac{\partial V_2}{\partial M_1} + \ldots \\ \frac{\partial F}{\partial M_2} &= \frac{\partial F}{\partial V_1} \frac{\partial V_1}{\partial M_2} + \frac{\partial F}{\partial V_2} \frac{\partial V_2}{\partial M_2} + \ldots \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{8}$$

Ex. 3. To find the m. s. e. of a side, a, in a triangle whose angles have been measured with the weights p_1, p_2, p_3, the base, b, being free from error.

The function equation is

$$F = a = b\,\frac{\sin A}{\sin B}$$

and the condition equation

$$A + B + C = 180 + \epsilon$$

Hence from Ex. 2, Art. 110, expressing A, B in terms of the observed values,

$$A = M_1 + \frac{u_1}{[u]}\left\{180 + \epsilon - (M_1 + M_2 + M_3)\right\}$$

$$B = M_2 + \frac{u_2}{[u]}\left\{180 + \epsilon - (M_1 + M_2 + M_3)\right\}$$

Now,

$$dF = \left(\frac{\partial F}{\partial A}\,\frac{\partial A}{\partial M_1} + \frac{\partial F}{\partial B}\,\frac{\partial B}{\partial M_1}\right)v_1$$
$$+ \left(\frac{\partial F}{\partial A}\,\frac{\partial A}{\partial M_2} + \frac{\partial F}{\partial B}\,\frac{\partial B}{\partial M_2}\right)v_2 + \left(\frac{\partial F}{\partial A}\,\frac{\partial A}{\partial M_3} + \frac{\partial F}{\partial B}\,\frac{\partial B}{\partial M_3}\right)v_3$$

$$= a \sin 1''\left(\left\{\left(1 - \frac{u_1}{[u]}\right)\cot A + \frac{u_2}{[u]}\cot B\right\}v_1\right.$$
$$\left. + \left\{-\frac{u_1}{[u]}\cot A - \left(1 - \frac{u_2}{[u]}\right)\cot B\right\}v_2 + \left\{-\frac{u_1}{[u]}\cot A + \frac{u_2}{[u]}\cot B\right\}v_3\right)$$

Therefore

$$u_F = a^2 \sin^2 1''\left(\left\{\left(1 - \frac{u_1}{[u]}\right)\cot A + \frac{u_2}{[u]}\cot B\right\}^2 u_1\right.$$
$$\left. + \left\{\frac{u_1}{[u]}\cot A + \left(1 - \frac{u_2}{[u]}\right)\cot B\right\}^2 u_2 + \left\{\frac{u_1}{[u]}\cot A - \frac{u_2}{[u]}\cot B\right\}^2 u_3\right)$$

$$= a^2 \sin^2 1''\left\{\left(u_1 - \frac{u_1^2}{[u]}\right)\cot^2 A + \left(u_2 - \frac{u_2^2}{[u]}\right)\cot^2 B + \frac{2u_1u_2}{[u]}\cot A \cot B\right\}$$

and
$$\mu_F = \mu\sqrt{u_F}$$

where μ is the m. s. e. of a single observation.

If the weights p_1, p_2, p_3 are each equal to unity, this reduces to

$$\mu_F^2 = \tfrac{2}{3}\,a^2 \sin^2 1''\,\mu^2\,(\cot^2 A + \cot^2 B + \cot A \cot B)$$

and if the triangle is equilateral,

$$\mu_F^2 = \tfrac{2}{3}\,a^2 \sin^2 1''\,\mu^2$$

Also, if the base, instead of being considered exact, had the m. s. e. μ_b, the expressions for μ_F^2 would be increased by $\frac{a^2}{b^2}\mu_b^2$ and μ_b^2 respectively.

It is, however, usually much more convenient in practice to use the method of correlates.

Let the function, reduced to the linear form, be written

$$dF = f'v_1 + f''v_2 + \ldots \tag{9}$$

This is conditioned by the n_c equations, also in the linear form,

$$\begin{aligned} a'v_1 + a''v_2 + \ldots - l' &= 0 \\ b'v_1 + b''v_2 + \ldots - l'' &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \end{aligned} \tag{10}$$

with

$$[pvv] = \text{a minimum}.$$

Referring to the principle of Art. 110, we see that by using correlates k', k'', . . ., and determining them properly, we can express the function in terms of the quantities v_1, v_2, . . . v_n as if independent; that is,

$$dF = (f' - a'k' - b'k'' - \ldots)\,v_1 \\ \qquad + (f'' - a''k' - b''k'' - \ldots)v_2 + \ldots \tag{11}$$

and, therefore,

$$u_F = (f' - a'k' - b'k'' - \ldots)^2 u_1 \\ \qquad + (f'' - a''k' - b''k'' - \ldots)^2 u_2 + \ldots \tag{12}$$

It remains to determine k', k'', . . . Now, when the most probable values of the corrections v_1, v_2, . . . v_n are substituted in the value of the function dF, this function must have its most probable value, and, therefore, its maximum weight. We may, therefore, determine the correlates k from the condition that the weight of dF is a maximum; that is, that u_F is a minimum. Differentiate, then, u_F with respect to k', k'', . . . as independent variables, and we have the equations

$$\begin{aligned} [uaa]k' + [uab]k'' + \ldots &= [uaf] \\ [uab]k' + [ubb]k'' + \ldots &= [ubf] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \end{aligned} \tag{13}$$

from which k', k'', . . . are found.

These equations being precisely of the form of ordinary normal equations, it follows, as in (c) and (d), Art. 100, that

$$u_F = [uff] - [uaf]k' - [ubf]k'' - \ldots \tag{14}$$

or

$$u_F = [uff] - \frac{[uaf]^2}{[uaa]} - \frac{[ubf.1]^2}{[ubb.1]} - \ldots \tag{15}$$

The form of the last expression for u_F shows that it may be found by means of the following scheme, in which $[uaf]$, $[ubf]$, . . . are added as an extra column in the solution of the correlate normal equations (13), in the manner shown in Art. 100. For three correlates the scheme would be

k'	k''	k'''	
$[uaa]$	$[uab]$	$[uac]$	$[uaf]$
	$[ubb]$	$[ubc]$	$[ubf]$
		$[ucc]$	$[ucf]$
			$[uff]$
	$[ubb.1]$	$[ubc.1]$	$[ubf.1]$
		$[ucc.1]$	$[ucf.1]$
			$[uff.1]$
		$[ucc.2]$	$[ucf.2]$
			$[uff.2]$
			$[uff.3]$
			$= u_F$

Ex. 4. To find the weight of the angle PSB in Ex. 1, Art. 109.

Here

$$dF = -v_1 + v_3$$

$$\therefore \quad f_1 = -1, \quad f_2 = 0, \quad f_3 = +1$$

From the condition equations

$$a' = +1 \qquad b'' = +1$$
$$a''' = -1 \qquad b''' = -1$$
$$a'''' = +1 \qquad b''''' = +1$$

$$\therefore \quad [uaf] = \tfrac{1}{5} \times -1 + \tfrac{1}{4} \times -1 = -0.45$$
$$[ubf] = -\tfrac{1}{4}$$
$$[ucf] = +0.45$$

The correlate normal equations with the extra column for finding u_F:

k'	k''	l		
$+0.5929$	$+0.2500$	-0.7600	$-0.4500 = [uaf]$	
$+0.2500$	$+0.6429$	-1.6600	$+0.2500 = [ubf]$	$+0.4500 = [uff]$
$+1$	$+0.4217$	-1.2818	-0.7590	
	$+0.5375$	-1.3395	$-0.0602 = [ubf.1]$	$+0.3416$
				$+0.1084 = [uff.1]$
	$+1$	-2.492	-0.1120	$+0.0067$
				$+0.1017 = [uff.2] = u_F$

Also

$$\mu_F = \mu \sqrt{u_F}$$
$$= 1.47 \sqrt{0.1017} \text{ from Ex. 2.}$$
$$= 0''.47$$

as before.

Ex. 5. To find the weight and m. s. e. of the adjusted value of an angle of a triangle when all three angles are measured, their weights being p_1, p_2, p_3 respectively.

The function is

$$dF = v_1$$

and the condition equation

$$v_1 + v_2 + v_3 = l$$

Hence from (15)

$$u_F = u_1 - \frac{u_1^2}{[u]}$$
$$= \frac{u_1(u_2 + u_3)}{[u]}$$

Also
$$\mu_F = \mu \sqrt{u_F}$$
$$= \frac{l}{\sqrt{[u]}}\sqrt{\frac{u_1(u_2+u_3)}{[u]}} \quad \text{(See Ex. 1.)}$$
$$= \frac{l}{[u]}\sqrt{u_1(u_2+u_3)}$$

The weight of an angle before adjustment is to the weight after adjustment, as

$$\frac{1}{u_1} : \frac{[u]}{u_1(u_2 + u_3)}$$

or
$$u_2 + u_3 : [u]$$

If $p_1 = p_2 = p_3 = 1$, the weights are as 2 : 3. This result is independent of the magnitude of the angle. It therefore applies to any problem in which the condition to be satisfied is that the sum of two quantities shall be equal to a third, or in which the sum of all three is equal to a constant. For other solutions see Ex. 2, Art. 109.

Ex. 6. If n angles measured at a station close the horizon, find the weight of the adjusted value of any one of them.

[The solution is exactly as in the preceding example.

The weight of V_1, for instance, is found from

$$u_{V_1} = \frac{u_1\{[u] - u_1\}}{[u]}$$

If the weights $p_1, p_2, \ldots$ are all equal to one another, the weight of an angle after adjustment is to its weight before adjustment as

$$n : n - 1 \quad]$$

Ex. 7. Show that the weight of the sum of the adjusted angles of a triangle is infinite.

[Sum = 180 + ε, a fixed quantity.

$$\therefore \quad \text{m. s. e.} = 0, \text{ and weight} = \infty$$

or otherwise

$$v_1 + v_2 + v_3 = l$$
$$a' = a'' = a''' = 1$$
$$\therefore \quad u_F = 3 - \frac{3^2}{3} = 0 \quad]$$

Ex. 8. In the "longitude triangle" Brest, Greenwich, Paris, as determined by the U. S. Coast Survey in 1872, the observed values were

	m. s.		
Brest–Greenwich,	17 57.154	weight	10
Greenwich–Paris,	9 21.120	"	7
Brest–Paris,	27 18.190	"	9

Show that the most probable values are

m. s.		
17 57.130	weight	14
9 21.086	"	12
27 18.216	"	13

Ex. 9. To find the weight of a side in a chain of triangles, all of the angles of each triangle having been equally well measured and the base being free from error.

Let b be the measured value of the base, and let $a_1, a_2, \ldots a_n$ be the sides of continuation in order as computed from b; a_n being the side whose weight is required.

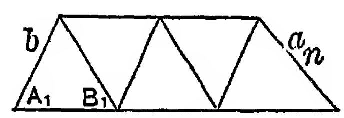


Fig. 12

If $A_1, B_1, A_2, B_2, \ldots$ are the measured values of the angles used in computing a_n from b, the angles $A_1, A_2, \ldots$ being opposite to the sides of continuation, then

$$\frac{a_1}{b} = \frac{\sin A_1}{\sin B_1}, \quad \frac{a_2}{a_1} = \frac{\sin A_2}{\sin B_2}, \quad \ldots, \quad \frac{a_n}{a_{n-1}} = \frac{\sin A_n}{\sin B_n}$$

Hence by multiplying these expressions together,

$$a_n = b \frac{\sin A_1}{\sin B_1} \frac{\sin A_2}{\sin B_2} \cdots \frac{\sin A_n}{\sin B_n} \tag{1}$$

We may now proceed in two ways.

(a) Differentiating directly,

$$da_n = a_n \sin 1'' \left[\cot A\,(A) - \cot B\,(B)\right]$$

where (A), (B), ... denote the corrections to A, B, ...

[In a chain of triangles it is convenient to use the notation (A), (B) ... for $v_1, v_2, \ldots$, the parentheses indicating corrections.]

The condition equations, from the closure of the triangles, are

$$\begin{aligned}(A_1) + (B_1) + (C_1) &= l' \\ (A_2) + (B_2) + (C_2) &= l'' \\ \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot\end{aligned} \tag{2}$$

Substituting in Eq. 15,

$$u_{a_n} = \tfrac{2}{3}\, a_n^2 \sin^2 1'' \left[\cot^2 A + \cot^2 B + \cot A \cot B\right] \tag{3}$$

the result required.

If the triangles are equilateral this reduces to

$$u_{a_n} = \tfrac{2}{3}\, n b^2 \sin^2 1'' \tag{4}$$

Hence in a chain of equilateral triangles the weights of the sides decrease as we proceed from the base, b, through the successive triangles, inversely as the number of triangles passed over; that is, are as the fractions

$$\tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots$$

(b) Taking logs. of both members of Eq. 1 and differentiating,

$$d \log a_n = \frac{d}{dA_1} \log \sin A_1 (A_1) - \frac{d}{dB_1} \log \sin B_1 (B_1) + \ldots$$
$$= [\delta_A(A) - \delta_B(B)] \qquad (5)$$

or expanding the first member,

$$\delta_{a_n} da_n = [\delta_A (A) - \delta_B(B)] \qquad (6)$$

where δ_a is the tabular difference for one unit for the number a_n, and δ_A, δ_B are the logarithmic differences corresponding to $1''$ for the angles A, B in a table of log. sines. (See Art. 7.)

Hence attending to the condition equations 2, we have from (15) for Eq. 5,

$$u_{\log a_n} = \tfrac{2}{3}[\delta_A{}^2 + \delta_A \delta_B + \delta_B{}^2]$$

and for Eq. 6,

$$u_{a_n} = \tfrac{2}{3} \frac{1}{\delta_{a_n}{}^2}[\delta_A{}^2 + \delta_A \delta_B + \delta_B{}^2]$$

as giving the weight of the logarithm of the side and the weight of the side respectively.

Of the two forms (a) and (b), the logarithmic is in general the most convenient in practice.

Ex. 10. From a base AB ($= b$) proceeds a chain of equilateral triangles, all of the angles being equally well measured, and the sides BC, CD, . . . being in the same straight line. Find the m. s. e. of the line BN, which is n times the base.

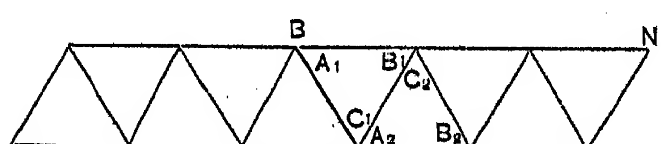


Fig. 13

Take first the simple case of $n = 2$.

$$F = BN = b\frac{\sin C_1}{\sin B_1} + b\frac{\sin A_1 \sin A_2 \sin C_3}{\sin B_1 \sin B_2 \sin B_3}$$

$$\therefore dF = \{\cot A_1 (A_1) - 2 \cot B_1 (B_1) + \cot C_1 (C_1) + \cot A_2 (A_2) - \cot B_2 (B_2) - \cot B_3 (B_3) + \cot C_3 (C_3)\} b \sin 1''$$

Also, we have the condition equations

$$(A_1) + (B_1) + (C_1) = l'$$
$$(A_2) + (B_2) + (C_2) = l''$$
$$(A_3) + (B_3) + (C_3) = l'''$$

Hence

$$
\begin{aligned}
[aa] &= 3 \qquad [af] = 0 \\
[bb.1] &= 3 \qquad [bf.1] = 0 \\
[cc.2] &= 3 \qquad [cf.2] = 0
\end{aligned}
$$

$$
\begin{aligned}
[ff] &= (\cot^2 A_1 + 4\cot^2 B_1 + \cot^2 C_1 + \cot^2 A_2 + \cot^2 B_2 + \cot^2 B_3 + \cot^2 C_3) b^2 \sin^2 1'' \\
&= \tfrac{10}{3} b^2 \sin^2 1'' \qquad \text{since } \cot^2 60 = \tfrac{1}{3}
\end{aligned}
$$

Substituting in Eq. 18,

$$u_{BN}^2 = \tfrac{10}{3} b^2 \sin^2 1''$$

and therefore

$$u_{BN} = u \sqrt{\tfrac{10}{3}}\, b \sin 1''$$

where u is the m. s. e. of an observed angle.

Generally,

$$
\begin{aligned}
dF = (n - 1) \cot A_1 (A_1) - n \cot B_1 (B_1) + \cot C_1 (C_1) \\
+ (n - 1) \cot A_2 (A_2) - (n - 1) \cot B_2 (B_2) \\
- (n - 1) \cot B_3 (B_3) + \cot C_3 (C_3) \\
+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

and

$$u_{BN} = b^2 \sin^2 1'' \left(\frac{4n^3 - 3n^2 + 5n}{9} \right)$$

If the chain proceeds in the opposite direction until $AN' = BN$, then since $u_{AN'}^2 = u_{BN}^2$, and $NN' = 2bn$ approximately, we have

$$u_{NN'}^2 = u\, NN' \sin 1'' \sqrt{\frac{4n^2 - 3n + 5}{18n}}$$

If NN' is n times the base $\left(\text{putting } n = \frac{n}{2}\right)$

$$u_{NN'} = u\, NN' \sin 1'' \sqrt{\frac{2n^2 - 3n + 10}{18n}}$$

Hence it follows that in a chain of equilateral triangles where one base only is measured, it is better to place the base at the centre of the chain rather than at either end.

Ex. 11. If a chain of equilateral triangles proceeds from the base AB, as the chain in Ex. 10, but in the opposite direction, show that the m. s. e. of BN', which is n times the base, is

$$\frac{ub \sin 1''}{3} \sqrt{(4n^3 + 3n^2 + 5n)}$$

and if the chain proceeds also in the opposite direction until $AN' = BN'$, then if NN' be taken n times the base,

$$u_{NN'} = u\, NN' \sin 1'' \sqrt{\frac{2n^2 + 3n + 10}{18n}}$$

Ex. 12. If a chain of equilateral triangles proceeds from the base AB, which is in the same straight line as the derived side BN, show that the m. s. e. of AN, which is $(n+1)$ times the base b, is

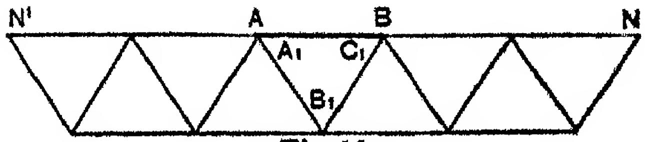


Fig. 14

$$\frac{\mu b}{3}\sin 1''\sqrt{4n^3+9n^2+5n}$$

$$\left[F = b + b\,\frac{\sin A_1 \sin A_2 \sin C_3}{\sin B_1 \sin B_2 \sin B_3} + b\,\frac{\sin A_1 \sin A_2 \sin A_3 \sin A_4 \sin C_5}{\sin B_1 \sin B_2 \sin B_3 \sin B_4 \sin B_5} + \qquad\right]$$

If also a chain of equilateral triangles proceeds from AB in the opposite direction to N', then if NN' is n times the base, show that the m. s. e. of NN' is

$$\mu NN' \sin 1''\sqrt{\frac{n^2+3n-4}{9n}}$$

Hence show that in computing a line NN', equal to n times the base AB, through a chain of equilateral triangles, the least loss of precision is with the form of Fig. 12.

Ex. 13. To find the m. s. e. of the altitude of a triangle, the base, b, being supposed free from error, and the reciprocal weights of the angles being u_1, u_2, u_3 respectively.

The function is

$$F = b\,\frac{\sin A \sin C}{\sin B}$$

$$\therefore\ dF = F\sin 1''\left\{\cot A\,(A) - \cot B\,(B) + \cot C\,(C)\right\}$$

Also the condition

$$(A)+(B)+(C)=l$$

Substituting in (15)

$$u_F = F^2\left\{u_1\cot^2 A + u_2\cot^2 B + u_3\cot^2 C - \frac{(u_1\cot A - u_2\cot B + u_3\cot C)^2}{[u]}\right\}\sin^2 1''$$

If

$$\angle A = \angle C, \text{ and } u_1 = u_2 = u_3 = \frac{1}{p}$$

then

$$u_F = \frac{2}{3}\,\frac{F^2}{p}\,\sin^2 1''\,\operatorname{cosec}^2 B$$

and

$$\mu_F = \frac{\mu b \sin 1''}{\sqrt{24p}}\operatorname{cosec}^2\frac{B}{2}$$

where μ is the m. s. e. of the angle corresponding to the unit of weight.

Ex. 14. If two similar isosceles triangles on opposite sides of the base AC are measured independently, thus forming a rhombus (vertices B, B'), then, taking the weight of each angle unity,

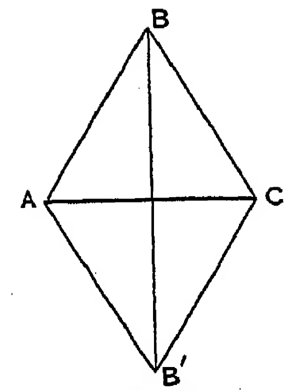


Fig. 15

$$\mu_{BB'} = \sqrt{2}\ \frac{\mu b \sin 1''}{\sqrt{24}} \operatorname{cosec}^2 \frac{B}{2}$$

and if BB' is n times the base b, then, since $\cot \frac{B}{2} = n$,

$$\mu_{BB'} = \frac{\mu BB' \sin 1''}{2\sqrt{3}}\left(n + \frac{1}{n}\right)$$

Caution.—If we solved for the rhombus directly it would not do to take

$$BB' = b \cot \frac{B}{2}$$

and then form $\mu_{BB'}$. The result would be $\sqrt{2}$ times too great. For as the triangles are measured independently, each half of BB' must be considered separately, so that we must use the form

$$BB' = \frac{b}{2}\left(\cot \frac{B}{2} + \cot \frac{B'}{2}\right)$$

with the condition equations

$$\begin{aligned}(A) + (B) + (C) &= l_1\\ (A') + (B') + (C') &= l_2\end{aligned}$$

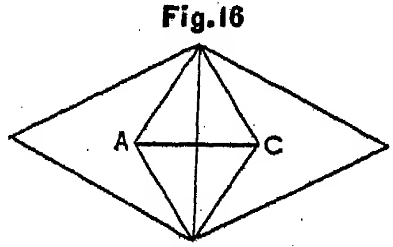
corresponding to the angles of the two triangles.

Ex. 15. If on a base, b, as diagonal two similar isosceles triangles are described, forming a rhombus, and on the other diagonal of this rhombus two triangles similar to the former are described, forming a second rhombus, and so on m times, required the m. s. e. of the last diagonal, all of the angles being equally well measured.

For the m^{th} diagonal d

$$F = d = \frac{b}{2^m}\left(\cot \frac{B_1}{2} + \cot \frac{B_2}{2}\right)\left(\cot \frac{B_3}{2} + \cot \frac{B_4}{2}\right) \ldots$$

Fig. 16



where $B_1, B_2, \ldots$ are the vertical angles in order.

Now, as the triangles are all similar,

$$B_1 = B_2 = B_3 = \ldots = B_{2m} = B \text{ suppose.}$$

Hence

$$f' = f'' = \ldots = \frac{b}{4} \cot^{m-1} \frac{B}{2} \operatorname{cosec}^2 \frac{B}{2} \sin 1''$$

and

$$u_d = 2m\left\{\left(\frac{b^2}{16}\cot^{2m-2}\frac{B}{2}\ \operatorname{cosec}^4\frac{B}{2}\right) - \frac{1}{3}\left(\frac{b}{4}\cot^{m-1}\frac{B}{2}\operatorname{cosec}^2\frac{B}{2}\right)^2\right\}\sin^2 1''$$

$$= \frac{mb^2}{12}\ \cot^{2m-2}\frac{B}{2}\operatorname{cosec}^4\frac{B}{2}\sin^2 1''$$

But

$$d = b\cot^m\frac{B}{2}$$

$$\therefore\ u_d = \frac{m}{3}d^2\operatorname{cosec}^2 B\sin^2 1''$$

If d is n times the base,

$$\mu_d = \mu d\sqrt{\frac{m}{3}}\ \frac{1+n^{\frac{2}{m}}}{2n^{\frac{1}{m}}}\ \sin 1''$$

For further development of this subject consult Helmert, *Studien über rationelle Vermessungen.* Leipzig, 1868.

Solution in Two Groups.

112. In geodetic work it often happens that the observed quantities are subject to a simple set of conditions which may be readily solved as observation equations by the method of independent unknowns, and are also subject to other conditions which are best solved by the method of correlates. The equations are thus divided into two groups for solution, and the complete solution, therefore, consists of two parts. The observation equations forming the first group are solved by themselves and give approximations to the final values of the unknowns. The corrections to these approximate values due to the second group are next found by solving this second group by the method of correlates.*

The merit of the method consists in utilizing the work expended in the solution of the first group in determining the additional corrections due to the second group. The

* The first exposition of this method was given by Bessel in the *Gradmessung in Ostpreussen.* The method of finding the precision of the adjusted values is due to Andrae, *Den Danske Gradmaaling*, vol. i. Very complete statements will be found in the introduction to *Die preussische Landestriangulation*, vol. i., Berlin, 1874; Ferrero, *Esposizione del metodo dei minimi quadrati*, Florence, 1876; Jordan, *Handbuch der Vermessungskunde*, Stuttgart, 1878.

solution is rigorous, and, being broken into two parts, is more easily managed than if all of the equations had been solved simultaneously.

Let the first group of equations be the observation equations, n in number and containing n_u unknowns ($n > n_u$),

$$\begin{aligned} a_1x + b_1y + \ldots - l_1 &= v_1 \quad \text{weight } p_1 \\ a_2x + b_2y + \ldots - l_2 &= v_2 \quad \text{"} \quad p_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{1}$$

and the second group the condition equations, n_c in number, involving the same unknowns ($n_c < n_u$),

$$\begin{aligned} a'x + a''y + \ldots - l' &= 0 \\ b'x + b''y + \ldots - l'' &= 0 \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{2}$$

The most probable values of the unknowns $x, y, \ldots$ are those which are given by the relation

$$[pvv] = \text{a minimum}. \tag{3}$$

It is required to find them.

The value of an unknown is found in two parts, the first, $(x), (y), \ldots$, arising from the observation equations, and the second, $(1), (2), \ldots$, arising from the condition equations, thus:

$$\begin{aligned} x &= (x) + (1) \\ y &= (y) + (2) \\ \cdot \quad \cdot & \quad \cdot \quad \cdot \end{aligned} \tag{4}$$

Now, overlooking for the present the condition equations and taking the observation equations only, $(x), (y), \ldots$ would be found by solving these equations in the usual way. We have, therefore, reducing all to weight unity for convenience in writing, the normal equations

$$\begin{aligned} [aa](x) + [ab](y) + \ldots &= [al] \\ [ab](x) + [bb](y) + \ldots &= [bl] \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{5}$$

The solution of these equations gives (see Art. 97)

$$(x) = [\alpha\alpha][al] + [\alpha\beta][bl] + \ldots$$
$$(y) = [\alpha\beta][al] + [\beta\beta][bl] + \ldots \qquad (6)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence (x), (y), . . . are known.

To find the condition corrections (1), (2), . . . , eliminate v_1, v_2, . . . v_n by substituting in the minimum equation, which then becomes

$$\begin{aligned}[aa]xx + 2[ab]xy + \ldots - 2[al]x& \\ + [bb]yy + \ldots - 2[bl]x& \qquad (7) \\ \cdot \quad \cdot \quad \cdot \quad \cdot& \\ + [ll] = \text{a min.}&\end{aligned}$$

This equation is conditioned by equations 2. Thus the solution is reduced to that already carried out in Art. 110.

Calling I, II, . . . the correlates of equations 2, we have the correlate equations

$$[aa]x + [ab]y + \ldots - [al] = a'I + b'II + \ldots$$
$$[ab]x + [bb]y + \ldots - [bl] = a''I + b''II + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

These equations, taken with (4) and (5), give the relations

$$[aa](1) + [ab](2) + \ldots = a'I + b'\,II + \ldots = \boxed{1} \text{ suppose}$$
$$[ab](1) + [bb](2) + \ldots = a''I + b''\,II + \ldots = \boxed{2} \quad \text{"} \qquad (8)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

which being of the same form as (5), their solution gives

$$(1) = [\alpha\alpha]\,\boxed{1} + [\alpha\beta]\,\boxed{2} + \ldots$$
$$(2) = [\alpha\beta]\,\boxed{1} + [\beta\beta]\,\boxed{2} + \ldots \qquad (9)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

or substituting for $\boxed{1}$, $\boxed{2}$, . . . their values from (8),

$$(1) = \text{A}'I + \text{B}'II + \text{C}'III + \ldots$$
$$(2) = \text{A}''I + \text{B}''II + \text{C}''III + \ldots \qquad (10)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where

$$\begin{aligned} \mathrm{A}' &= [\alpha\alpha]a' + [\alpha\beta]a'' + \ldots \\ \mathrm{B}' &= [\alpha\alpha]b' + [\alpha\beta]b'' + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{11}$$

and are known quantities.

We have, therefore, expressed the corrections (1), (2), . . . in terms of the unknown correlates I, II, . . . It remains now to find these correlates.

Substituting for $x, y,$ . . . their values from (4) in the condition equations, and

$$\begin{aligned} a'(1) + a''(2) + \ldots &= l_0' \\ b'(1) + b''(2) + \ldots &= l_0'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned} \tag{12}$$

where

$$\begin{aligned} l_0' &= l' - a'(x) - a''(y) - \ldots \\ l_0'' &= l'' - b'(x) - b''(y) - \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{13}$$

and are, therefore, known quantities, since (x), (y), . . . are known.

Substitute the values of (1), (2), . . ., from (10) in (12), and we have the correlate normal equations

$$\begin{aligned} [a\mathrm{A}]\, I + [a\mathrm{B}]\, II + \ldots &= l_0' \\ [a\mathrm{B}]\, I + [b\mathrm{B}]\, II + \ldots &= l_0'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \end{aligned} \tag{14}$$

where

$$\begin{aligned} [a\mathrm{A}] = [\alpha\alpha]a'a' &+ [\alpha\beta]a'a'' + \ldots \\ &+ [\alpha\beta]a'a'' + [\beta\beta]a''a'' + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{15}$$

$$\text{etc.} = \text{etc.}$$

The solution of these equations gives the correlates I, II, . . . Hence the corrections (1), (2), . . . are known. Also, since (x), (y), . . . have been found from (6), the total corrections $x, y,$ . . . are known.

113. In carrying the preceding solution into practice the following order of procedure will be found convenient:

(a) The formation and solution of the observation equations (1).

The partially adjusted resulting values (x), (y), . . . are now to be used.

(b) The formation of the condition equations (12).

$$a'(1) + a''(2) + \ldots = l_0'$$
$$b'(1) + b''(2) + \ldots = l_0''$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(c) The formation of the weight equations (9). They are at once written down from the general solution of the observation equations in (a), and are

$$(1) = [\alpha\alpha]\,\boxed{1} + [\alpha\beta]\,\boxed{2} + \ldots$$
$$(2) = [\alpha\beta]\,\boxed{1} + [\beta\beta]\,\boxed{2} + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(d) The formation of the correlate equations (8).

$$\boxed{1} = a'\,I + b'\,II + \ldots$$
$$\boxed{2} = a''\,I + b''\,II + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(e) The expression of the corrections in terms of the correlates by substituting from (d) in (c).

$$(1) = \mathrm{A}'\,I + \mathrm{B}'\,II + \ldots$$
$$(2) = \mathrm{A}''I + \mathrm{B}''II + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(f) The formation of the normal equations by substituting from (e) in (b). They are,

$$[a\mathrm{A}]\,I + [a\mathrm{B}]\,II + \ldots = l_0'$$
$$[a\mathrm{B}]\,I + [b\mathrm{B}]\,II + \ldots = l_0''$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(g) The determination of the corrections by substituting the values of the correlates in (e).

114. **To Find the Precision of the Adjusted Values or of any Function of them.**

(a) First find μ, the m. s. e. of an observation of weight unity.

We have (Art. 111)

$$\mu^2 = \frac{[vv]}{\text{number of conditions}}$$
$$= \frac{[vv]}{(n - n_u) + n_c}$$

since $n - n_u$ is the number of conditions in the observation equations, and n_c the number in the condition equations.

To find $[vv]$. From the first observation equation

$$\begin{aligned} v_1 &= a_1x + b_1y + \ldots \quad - l_1 \\ &= a_1(x) + b_1(y) + \ldots - l_1 + a_1(1) + b_1(2) + \ldots \\ &= v_1^0 + a_1(1) + b_1(2) + \ldots \end{aligned}$$

Similarly

$$v_2 = v_2^0 + a_2(1) + b_2(2) + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where

$$\begin{aligned} v_1^0 &= a_1(x) + b_1(y) + \ldots - l_1 \\ v_2^0 &= a_2(x) + b_2(y) + \ldots - l_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

that is, v_1^0, v_2^0, . . . are the residuals arising from taking the observation equations only.

Attending to Eq. 5, p. 239, it follows evidently that

$$[av^0] = 0 \quad [bv^0] = 0, \ldots$$

Square the residuals v_1, v_2, . . . and add, then

$$\begin{aligned} [vv] &= [v^0v^0] + [\{a(1) + b(2) + \ldots\}^2] \\ &= [v^0v^0] + [ww] \text{ suppose.} \end{aligned}$$

The total sum $[vv]$ may therefore be found in two parts, one from squaring the residuals of the observation equations, and the other from the corrections (1), (2), . . .

We proceed to put $[ww]$ in a more convenient shape for computation.

$$\begin{aligned}[ww] &= [\{a(1) + b(2) + \ldots\}^2] \\ &= (1)\{[aa](1) + [ab](2) + \ldots\} \\ &\quad + (2)\{[ab](1) + [bb](2) + \ldots\} \\ &\quad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= (1)\,\boxed{1} + (2)\,\boxed{2} + \ldots\end{aligned}$$

from Eq. 8, p. 240.

Substitute for (1), $\boxed{1}$, (2), . . . their values from equations 8 and 10, and expand; then

$$\begin{aligned}[ww] &= \{[\overline{a\mathrm{A}}]\,I + [\overline{a\mathrm{B}}]\,II + \ldots\}I \\ &\quad + \{[a\mathrm{B}]\,I + [\overline{b\mathrm{B}}]\,II + \ldots\}II \\ &\quad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

which may be transformed, by means of Eq. 14, into the form

$$[ww] = l_0'I + l_0''II + \ldots$$

or, as in Art. 111, into the form

$$[ww] = \frac{(l_0')^2}{[a\mathrm{A}]} + \frac{(l_0''.1)^2}{[b\mathrm{B}.1]} + \frac{(l_0'''.2)^2}{[c\mathrm{C}.2]} + \ldots$$

These forms may be readily computed as in Art. 100.

(b) Next find the weight of the given function of the adjusted values.

Let the function, reduced to the linear form, be

$$dF = g_1x + g_2y + \ldots \tag{16}$$

where g_1, g_2, . . . are known quantities.

Put for x, y, . . . their values $(x)+(1)$, $(y)+(2)$, . . . and

$$dF = g_1(x) + g_2(y) + \ldots + g_1(1) + g_2(2) + \ldots$$

Put for (1), (2), . . . their values from (10), and

$$dF = g_1(x) + g_2(y) + \ldots + [g\mathrm{A}]I + [g\mathrm{B}]II + \ldots \tag{17}$$

where I, II, . . . are found from the equations

$$\begin{aligned}
&[a\text{A}]I + [a\text{B}]II + \ldots - l_0' = 0 \\
&[a\text{B}]I + [b\text{B}]II + \ldots - l_0'' = 0 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Using the multipliers k_1, k_2, . . . in order to eliminate I, II, . . ., we have, as in Art. 111,

$$\begin{aligned}
dF = g_1(x) + g_2(y) + \ldots + l_0'k_1 + l_0''k_2 + \ldots& \\
+ \{[g\text{A}] - [a\text{A}]k_1 - [a\text{B}]k_2 - \ldots\}I& \\
+ \{[g\text{B}] - [a\text{B}]k_1 - [b\text{B}]k_2 - \ldots\}II& \\
+ \ldots& \qquad (18)
\end{aligned}$$

We may determine k_1, k_2, . . . so as to cause the coefficients of I, II, . . . to vanish; that is, so as to satisfy the equations

$$\begin{aligned}
&[a\text{A}]k_1 + [a\text{B}]k_2 + \ldots = [g\text{A}] \\
&[a\text{B}]k_1 + [b\text{B}]k_2 + \ldots = [g\text{B}] \qquad (19) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

and then we shall have

$$dF = g_1(x) + g_2(y) + \ldots + l_0'k_1 + l_0''k_2 + \ldots$$

Substitute for l_0', l_0'', . . . from (13), and

$$dF = [lk] + G_1(x) + G_2(y) + \ldots \qquad (20)$$

where

$$\begin{aligned}
&G_1 = g_1 - a'k_1 - b'k_2 - \ldots \\
&G_2 = g_2 - a''k_1 - b''k_2 - \ldots \qquad (21) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

We have thus expressed the function in terms of (x), (y), . . . and known quantities.

Now, since (x), (y), . . . are not independent, but are connected by the equations

$$\begin{aligned}
&[aa](x) + [ab](y) + \ldots = [al] \\
&[ab](x) + [bb](y) + \ldots = [bl] \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

the problem is reduced to that already solved in Art. 101.

If, therefore, u_F is the reciprocal of the required weight,

$$u_F = [GQ] \tag{22}$$

where

$$\begin{aligned} Q_1 &= [\alpha\alpha]G_1 + [\alpha\beta]G_2 + \ldots \\ Q_2 &= [\alpha\beta]G_1 + [\beta\beta]G_2 + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{23}$$

the quantities $[\alpha\alpha]$, $[\alpha\beta]$, . . . being as in the weight equations 9.

Putting for G_1, G_2, . . . their values from (21) in these equations, and attending to (11), we find

$$\begin{aligned} Q_1 &= q_1 - \mathrm{A}'k_1 - \mathrm{B}'k_2 - \ldots \\ Q_2 &= q_2 - \mathrm{A}''k_1 - \mathrm{B}''k_2 - \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{24}$$

where

$$\begin{aligned} q_1 &= [\alpha\alpha]g_1 + [\alpha\beta]g_2 + \ldots \\ q_2 &= [\alpha\beta]g_1 + [\beta\beta]g_2 + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{25}$$

Substituting in (22) for G_1, G_2, . . . Q_1, Q_2, . . . their values from (21) and (24),

$$\begin{aligned} [GQ] = [gq] &- [g\mathrm{A}]k_1 - [g\mathrm{B}]k_2 - \ldots \\ &- [aq]k_1 - [bq]k_2 - \ldots \\ &+ \{[a\mathrm{A}]k_1 + [a\mathrm{B}]k_2 + \ldots\}k_1 \\ &+ \{[a\mathrm{B}]k_1 + [b\mathrm{B}]k_2 + \ldots\}k_2 \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

But from (11) and (25)

$$[aq] = [g\mathrm{A}], \; [bq] = [g\mathrm{B}], \ldots$$

Hence, attending to (19), the above expression reduces to

$$[GQ] = [gq] - [g\mathrm{A}]k_1 - [g\mathrm{B}]k_2 - \ldots$$

or to

$$[GQ] = [gq] - \frac{[g\mathrm{A}]^2}{[a\mathrm{A}]} - \frac{[g\mathrm{B}.1]^2}{[b\mathrm{B}.1]} - \frac{[g\mathrm{C}.2]^2}{[c\mathrm{C}.2]} - \ldots$$

To compute $[gq]$. Multiply each of equations 25, in order, by $g_1, g_2, \ldots$, and add, and

$$\begin{aligned}[gq] = [\alpha\alpha]g_1g_1 + 2[\alpha\beta]g_1g_2 &+ \ldots \\ + [\beta\beta]g_2g_2 &+ \ldots \\ &+ \ldots\end{aligned}$$

where $[\alpha\alpha]$, $[\alpha\beta]$, . . . may be taken from the weight equations.

The remaining terms of the second form of $[GQ]$ may be found from the solution of the normal equations, as shown in Art. 111.

Solution by Successive Approximation.

115. This method of solution (due to Gauss) is of the greatest importance in adjustments involving many conditions. It may be stated as follows:

The condition equations may be divided into groups, and the groups solved in any order we please. Each successive group will give corrections to the values furnished by the preceding groups, and the corrected values will be closer and closer approximations to the most probable values which would be found from the simultaneous solution of all the groups.

For suppose we have the condition equations

$$\begin{aligned}a'v_1 + a''v_2 + \ldots &= l_a \\ b'v_1 + b''v_2 + \ldots &= l_b \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\ h'v_1 + h''v_2 + \ldots &= l_h \\ k'v_1 + k''v_2 + \ldots &= l_k \\ \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot\end{aligned}$$

with

$$[pv^2] = \text{a min.}$$

Let v_1', v_2', . . . be the values of v_1, v_2, . . . obtained from solving the first group alone; that is, from

$$\begin{array}{l} a'v_1' + a''v_2' + \ldots = l_a \\ b'v_1' + b''v_2' + \ldots = l_b \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

$$[pv'^2] = \text{a min.}$$

If now (v_1'), (v_2'), . . . are the corrections to these values resulting from the remaining equations, then since

$$\begin{aligned} [pv^2] &= [p\{v' + (v')\}^2] \\ &= [pv'^2] + [p(v')^2] \end{aligned}$$

the condition equations are reduced to

$$\begin{array}{l} a'(v_1') + a''(v_2') + \ldots = l_a' \\ b'(v_1') + b''(v_2') + \ldots = l_b' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ h'(v_1') + h''(v_2') + \ldots = l_h' \\ k'(v_1') + k''(v_2') + \ldots = l_k' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

with

$$[p(v')^2] = \text{a min.},$$

and the values of (v') found from the simultaneous solution of these equations, added to the values of v' found from the solution of the first set, would be equal to the value of v found directly.

Similarly if v_1'', v_2'', . . . be the values of (v') obtained by solving the second set alone, and (v_1''), (v_2''), . . . be the corrections to these values resulting from the remaining equations, then since

$$[pv^2] = [pv'^2] + [pv''^2] + [p(v'')^2]$$

the condition equations are reduced to

$$\begin{array}{l} a'(v_1'') + a''(v_2'') + \ldots = l_a'' \\ b'(v_1'') + b''(v_2'') + \ldots = l_a'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ h'(v_1'') + h''(v_2'') + \ldots = l_h'' \\ k'(v_1'') + k''(v_2'') + \ldots = l_k'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

with

$$[p(v'')^2] = \text{a min.}$$

The quantities $[pv'^2]$, $[pv''^2]$, . . . being positive, the minimum equation is reduced with the solution of each set, and thus we gradually approach the most probable set of values. Beginning with the first set a second time, and solving through again, we should reduce the minimum equation still farther, and by continuing the process we shall finally reach the same result as that obtained from the rigorous solution. In practice the first approximation is in general close enough.

It is plain that the most probable values can be found after any approximation by solving simultaneously the whole of the groups, using the values already found as approximations to these most probable values.

Examples will be found in the next chapter.

CHAPTER VI.

APPLICATION TO THE ADJUSTMENT OF A TRIANGULATION.

116. The adjustment of the measured angles of a triangulation net is a special case of the problem discussed in the preceding chapters. We assume the reader to be acquainted with the construction and method of handling of instruments used in measuring horizontal angles, and shall confine ourselves to the methods of adjusting the measured values of the angles.

117. For clearness we will explain in some detail the preliminary work necessary for the formation of the condition equations. In a triangulation there must be one measured base at least, as AB. Starting from this base and measuring the angles CAB, ABC, we may compute the sides AC, BC by the ordinary rules of trigonometry. In plotting the figure the point C can be located in but one way, as only the measurements necessary for this purpose have been made.

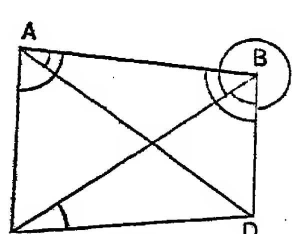


Fig. 17

Similarly, by measuring the angles CBD, DCB we may plot the position of the point D, and this can be done in but one way. If, however, the observer, while at A, had also read the angle DAB, then the point D could have been plotted in two ways, and we should find in almost all cases that the lines AD, BD, CD would not intersect in the same point. In other words, in computing the length of a side from the base we should find different values, according to the triangles through which we passed. Thus the value of CD computed from AB would not, in general, be the same if found from the triangles ABC, BCD, and from ABC, CAD.

If the blunt angle ABD had also been measured we should have another contradiction, arising from the non-satisfaction of the relation

$$DBC + CBA + ABD = 360^\circ$$

And not these contradictions only. For we have considered so far that in a triangle only two of the angles are measured. If in the first triangle, ABC, the third angle, BCA, were also measured, we know from spherical geometry that the three angles should satisfy the relation

$$CAB + ABC + BCA = 180^\circ + \text{sph. excess of triangle}$$

which the measured values will not do in general. A similar discrepancy may be expected in the other triangles.

In a triangulation net, then, with a single measured base, in which the sides are to be computed from this base through the intervening triangles, we conclude that the contradictions among the measured angles may be removed and a consistent figure obtained if the angles are adjusted so as to satisfy the two classes of conditions:

(1) Those arising at each station from the relations of the angles to one another at that station.

These are known as *local* conditions.

(2) Those arising from the geometrical relations necessary to form a closed figure.

(a) That the sum of the angles of each triangle in the figure should be equal to 180° increased by the spherical excess of the triangle.

(b) That the length of any side, as computed from the base, should be the same whatever route is chosen.

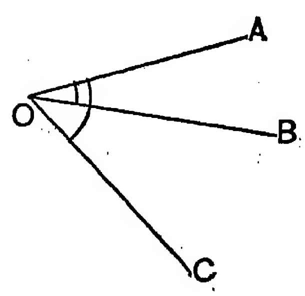
These are known as *general* conditions.

118. The number of conditions to be satisfied will depend on the measurements made. Each condition can be stated in the form of an equation in which the most probable values of the measured quantities are the unknowns. The number of equations being less than the number of

unknowns, an infinite number of solutions is possible. The problem before us is to select the most probable values from this infinite number of possible values.

The general statement of the method of solution is this: Adjust the angles so as to satisfy simultaneously the local and general conditions; that is, of all possible systems of corrections to the observed quantities which satisfy these conditions, to find that system which makes the sum of the squares of the corrections a minimum.

The form of the reduction depends on the methods employed in making the observations. These methods, in general terms, are as follows: Let O in the figure be the station occupied, and A, B, C signals sighted at. The angles AOB, BOC are required. By pointing at A and then at B we find the angle AOB. Point now at B and next at C, and we have the angle BOC. These two angles are *independent* of one another.



If, however, we had pointed at A, B, C in succession we should also have found the angles AOB, BOC, but they would not be independent of one another, as the reading to B enters into each.

In the first method of measurement, which is known as the method of independent angles, either a repeating or a non-repeating theodolite may be used; in the second, or method of directions, a non-repeating theodolite only.

The Method of Independent Angles.

119. As the case of independent angles is the simplest to reduce, we shall begin with it.

A distinction must be made between angles that are independently observed and angles which are independent in the sense that no condition exists between them. Thus at the station O, above, the angles AOB, BOC, AOC might be

observed independently of one another, but we should not call them independent angles, since the condition

$$AOC = AOB + BOC$$

must be satisfied between them. By independent angles, therefore, in the reduction, we mean those measured angles in terms of which all the measured angles can be expressed by means of the conditions connecting them. In the present case any two of the three angles AOB, BOC, AOC may be taken as independent, and the third angle would be dependent.

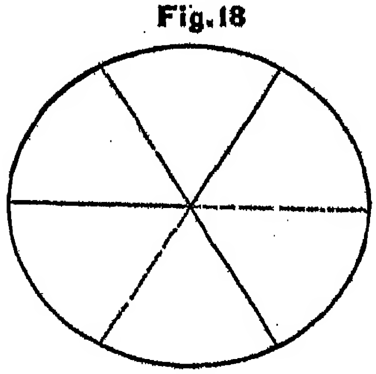
Angles may be measured independently either with a repeating or with a non-repeating theodolite. In primary work a non-repeating theodolite in which the graduated limb is read by microscopes furnished with micrometers is to be preferred. The method of reading an angle is as follows: The instrument, having been carefully adjusted, is directed to the left-hand signal and the micrometers read. It is then directed to the other signal and the micrometers again read. The difference between these readings is called a positive single result. The whole operation is repeated in reverse order; that is, beginning with the second signal and ending with the first, giving a negative single result. The mean of these two results is called a combined result, and is free from the error arising from uniform twisting of the post or tripod on which the instrument is placed, or from "twist of station," as it is called.

The telescope is next turned 180° in azimuth and then 180° in altitude, leaving the same pivots in the same wyes, and another combined result obtained. The mean of the two combined results is free from errors of the instrument arising from imperfect adjustments for collimation, from inequality in the heights of the wyes, and from inequality of the pivots.

The distinction between these two combined results is noted in the record by "telescope direct" and "telescope reverse."

Besides those mentioned there are two kinds of systematic error in measuring angles that deserve special attention. They are the errors arising from the regular or "periodic" errors of graduation of the horizontal limb of the instrument, and the error from the inclination of the limb itself to the horizon. The effects of the first may be got rid of by the method of observation, as follows: The reading of the limb on the first signal is changed (usually after each pair of combined results) by some aliquot part of the distance, or half-distance, between consecutive microscopes in case of two-microscope and three-microscope instruments respectively. Thus if n is the number of pairs of combined results desired, the changes would be $\frac{180}{n}$ and $\frac{60}{n}$ respectively with the instruments mentioned. The operation of reversal in case of a three-microscope instrument causes each microscope to fall at the middle of the opposite 120° space, the limb remaining unchanged. Thus if the full lines in Fig. 18 represent the positions of the microscopes with telescope direct, the dotted lines show their positions with telescope reverse. In this lies the greatest advantage of three microscopes over two, since with the latter, in reversing, the microscopes simply change places with each other, without reading on new portions of the limb.

Fig. 18



The error arising from want of level of the horizontal limb cannot be eliminated by the method of observation, but with the levels which accompany a good instrument, and with ordinary care, it will usually be less than 0".1. In case, however, of a signal having a high altitude above the horizon, the error from this source may be greater, and then special care should be taken in levelling. For an expression for its influence in any case see Chauvenet's *Astronomy*, Vol. II. Art. 211.

The observations should be made on at least two days

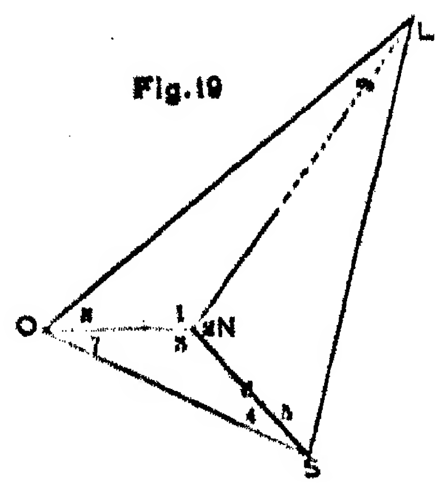
when conditions are favorable. Results obtained at different hours of a day are of more value than the same number of results obtained on different days at the same hour of the day. This is on account of variation in external conditions (direction of light, phase, distinctness, refraction, etc.)

120. We shall for illustration take the following example, making use of such parts of it from time to time as may belong to the subject in hand, and finally, after explaining the method of forming the condition equations, solve it in full.

In the triangulation of Lake Superior executed by the U. S. Engineers the following angles were measured in the quadrilateral N. Base, S. Base, Lester, Oncota.

LNO =	124° 09' 40".09	weight	2
SNL =	113° 30' 05".07	"	2
ONS =	122° 11' 15".61	"	14
NSO =	23° 08' 05".26	"	23
LSN =	47° 31' 20".41	"	6
LSO =	70° 39' 24".09	"	7
SON =	34° 40' 39".66	"	31
NOL =	43° 46' 26".40	"	1
OLS =	30° 53' 30".81	"	8

Fig. 19



These angles we shall denote by $M_1, M_2, \ldots, M_9$ respectively.

The length of the line N. Base — S. Base (Minnesota Point) is $6056^m.6$, and the latitudes of the four stations are approximately

N. Base, 46° 45'	Lester, 46° 52'
S. Base, 46° 43'	Oncota, 46° 45'

121. **The Local Adjustment.**—When in a system of triangulation the horizontal angles read at a station are adjusted for all of the conditions existing among them, then these angles are said to be locally adjusted.

From the considerations set forth in Art. 117 it is readily seen that at a station only two kinds of conditions are possible—

(a) that an angle can be formed from two or more others, and

(b) that the sum of the angles round the horizon should be equal to 360°.

The second of these is included in the first, and the method of adjustment may be stated in general terms as follows:

An inspection of the figure representing the angles at the station will show how all of the measured angles can be expressed in terms of a certain number of them which are independent of one another. These relations will give rise to condition equations, or *local equations*, as they are called, which may be solved as in Chapters IV. or V.

Thus, if $M_1, M_2, \ldots M_n$ denote the single measured angles, and $v_1, v_2, \ldots v_n$ their most probable corrections, then if any of the angles M_k, M_h can be formed from others, we have, by equating the measured and computed values, the local condition equations,

$$
\begin{aligned}
M_k + v_k &= M_1 + v_1 + M_2 + v_2 + \ldots \\
M_h + v_h &= M_1 + v_1 + M_2 + v_2 + \ldots \\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

or

$$
\begin{aligned}
v_1 + v_2 + \ldots - v_k &= l_k \text{ suppose} \\
v_1 + v_2 + \ldots - v_h &= l_h \quad \text{"} \\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

with

$$p_1 v_1^2 + p_2 v_2^2 + \ldots + p_k v_k^2 + p_h v_h^2 + \ldots + p_n v_n^2 = \text{a minimum}$$

where $p_1, p_2, \ldots p_n$ denote the weights of the angles.

The solution may be in general best carried out by the method of correlates, as in Chap. V.

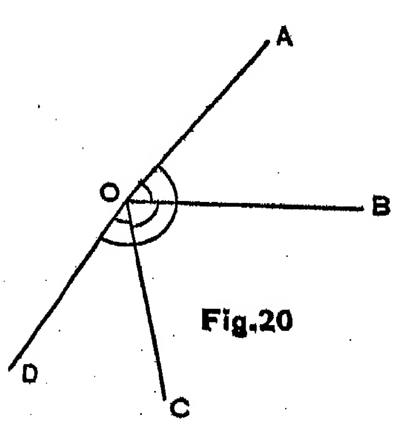


Fig.20

The following special cases are of frequent occurrence:

(1) At a station O the $n - 1$ single angles AOB, BOC, . . . are measured, and also the sum angle AOL, to find the adjusted values of the separate angles, all of the measured values being of the same weight.

The condition equation is

$$M_1 + v_1 + M_2 + v_2 + \ldots + M_{n-1} + v_{n-1} = M_n + v_n$$

or

$$v_1 + v_2 + \ldots + v_{n-1} - v_n = M_n - (M_1 + M_2 + \ldots + M_{n-1})$$
$$= l \text{ suppose,}$$

with

$$[v^2] = \text{a minimum.}$$

The solution gives (Art. 109 or 110)

$$v_1 = v_2 = \ldots = -v_n = \frac{l}{n}$$

that is, *the correction to each angle is* $\frac{1}{n}$ *of the excess of the sum angle over the sum of the single angles, and the sign of the correction to the sum angle is opposite to that of the single angles.*

(2) At a station O the n single angles AOB, BOC, . . . LOA are measured, thus closing the horizon, to find the adjusted values of the angles.

The condition equation is

$$v_1 + v_2 + \ldots + v_n = 360° - (M_1 + M_2 + \ldots + M_n)$$
$$= l \text{ suppose,}$$

with $$[pv^2] = \text{a minimum.}$$

The solution gives

$$v_1 = u_1 \frac{l}{[u]}$$
$$v_2 = u_2 \frac{l}{[u]}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

where $u_1 = \frac{1}{p_1}$, $u_2 = \frac{1}{p_2}$, . . ., and $[u] = \left[\frac{1}{p}\right]$.

If the weights are equal, then

$$v_1 = v_2 = \ldots = v_n = \frac{l}{n}$$

that is, *the correction to each angle is* $\frac{1}{n}$ *of the excess of* 360° *over the sum of the measured angles.*

Ex. 1. The angles at station N. Base close the horizon; required to adjust them.

We have (Art. 120)

$$\begin{aligned} M_1 + v_1 &= 124^\circ\ 09'\ 40''.69 + v_1 \quad \text{weight } 2 \\ M_2 + v_2 &= 113^\circ\ 39'\ 05''.07 + v_2 \quad \text{"} \quad 2 \\ M_3 + v_3 &= 122^\circ\ 11'\ 15''.61 + v_3 \quad \text{"} \quad 14 \end{aligned}$$

$$\begin{aligned} \text{Sum} &= 360^\circ\ 00'\ 01''.37 + v_1 + v_2 + v_3 \\ \text{Theoretical sum} &= 360^\circ\ 00'\ 00''.00 \end{aligned}$$

$\therefore$ Local equation is $\quad 0 = 1''.37 + v_1 + v_2 + v_3$

Hence (Ex. 2, Art. 110)

$$\begin{aligned} v_1 &= -\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{14}} \times 1.37 \\ &= -0''.64 \\ v_2 &= -0''.64 \\ v_3 &= -0''.09 \end{aligned}$$

and the adjusted angles are

$$\begin{aligned} &124^\circ\ 09'\ 40''.05 \\ &113^\circ\ 39'\ 04''.43 \\ &122^\circ\ 11'\ 15''.52 \\ \text{Check-sum} = &\ 360^\circ\ 00'\ 00''.00 \end{aligned}$$

Ex. 2. Precisely as in the preceding we may deduce at station South Base the local equation

$$0 = 1''.07 + v_4 + v_5 - v_6$$

and the adjusted angles

$$\begin{aligned} &23^\circ\ 08'\ 05''.13 \\ &47^\circ\ 31'\ 19''.91 \\ &70^\circ\ 39'\ 25''.04 \end{aligned}$$

122. *Number of Local Equations at a Station.*—If s stations are sighted at from a station that is occupied, the number of angles necessary to be measured to determine all of the angles that can be formed at the station occupied is $s - 1$. If, therefore, an additional angle were measured, its value could be determined in two ways: from the direct measurement and from the $s - 1$ measures. The contradiction in these two values would give rise to a local (condition) equation. If, therefore, n is the total number of angles measured at a station, the number of local equations, as indicated by the number of superfluous angles, is

$$n - s + 1.$$

123. **The General Adjustment.**—With a single measured base the number of conditions arising from the geometrical relations existing among the different parts of a triangulation net can be readily estimated. For if the net contains s stations, two are known, being the end points of the base, and $s - 2$ are to be found.

Now, two angles observed at the end points of the base will determine a third point; two more observed at the end points of a line joining any two of these points will determine a fourth point, and so on. Hence to determine the $s - 2$ points, $2(s - 2)$ angles are necessary. If, therefore, n is the total number of angles measured, the number of superfluous angles, that is, the number of conditions to be satisfied, is

$$n - 2(s - 2)$$

Ex. In a chain of triangles, if s is the number of stations, show that the number of conditions to be satisfied is $s - 2$; and in a chain of quadrilaterals, with both diagonals drawn, the number of conditions is $2s - 4$.

The equations arising from these conditions are divided into two classes, angle equations and side equations.

124. *The Angle Equations.*—The sum of the angles of a triangle drawn on a plane surface is equal to 180°. The sum of the angles of a spherical triangle exceeds 180° by the spherical excess (ε) of the triangle, which latter is found from the relation

$$\varepsilon = \frac{\text{area of triangle}}{R^2 \sin 1''}$$

R being the radius of the sphere.

From surveys carried on during the past two centuries the earth has been found to be spheroidal in form, and its dimensions have been determined within small limits. Now, a spheroidal triangle of moderate size may be computed as a spherical triangle on a tangent sphere whose radius is $\sqrt{\rho_1 \rho_2}$, where ρ_1, ρ_2 are the radii of curvature of the meridian and of the normal section to the meridian respectively at

the point corresponding to the mean of the latitudes φ of the triangle vertices.

Hence we may wrap our triangulation on the spheroid in question by conforming it to the spherical excess computed from the formula

$$\varepsilon \text{ (in seconds)} = \frac{ab \sin C}{2\rho_1 \rho_2 \sin 1''}$$

where a, b are two sides and C is the included angle of the triangle.

For convenience of computation we may write

$$\varepsilon = \lambda ab \sin C$$

when $\log \lambda$ may be tabulated for the argument φ. The following table is computed with Clarke's values of the elements of the terrestrial spheroid corresponding to latitudes from 10° to 70°. The metre is the unit of length to be used.

φ	$\log \lambda$	φ	$\log \lambda$	φ	$\log \lambda$
10°	1.40675	30°	1.40547	50°	1.40349
11°	672	31°	537	51°	339
12°	668	32°	528	52°	329
13°	663	33°	519	53°	319
14°	659	34°	509	54°	309
15°	1.40654	35°	1.40500	55°	1.40299
16°	649	36°	491	56°	289
17°	643	37°	481	57°	280
18°	637	38°	471	58°	271
19°	631	39°	461	59°	262
20°	1.40625	40°	1.40451	60°	1.40253
21°	618	41°	441	61°	244
22°	611	42°	431	62°	235
23°	604	43°	420	63°	226
24°	597	44°	410	64°	218
25°	1.40589	45°	1.40400	65°	1.40210
26°	581	46°	390	66°	202
27°	573	47°	380	67°	195
28°	564	48°	369	68°	188
29°	555	49°	359	69°	181
30°	1.40547	50°	1.40349	70°	1.40174

To find a, b, φ, a preliminary geodetic computation must first be made of the triangulation to be adjusted, starting from a base or from a known side. The values found from using the unadjusted angles will be close enough for this purpose. The latitudes need only be computed to the nearest minute.

A useful check of the excess results from the principle that the sums of the excesses of triangles that cover the same area should be equal. In our example the spherical excesses of the triangles *ONS*, *LSO* will be found to be $0''.05$ and $0''.37$ respectively.

In each single triangle, then, the condition required to wrap it on the spheroid, that is, that the sum of the three measured angles shall be equal to 180°, together with the spherical excess, gives a condition equation.* This is called an *angle equation*, or by some a *triangle equation*.

Ex. In the triangle N. Base, S. Base, Oneota, if v_3, v_4, v_7 denote the corrections to the three angles, we have for the most probable values

$$\begin{aligned} ONS &= 122^\circ\ 11'\ 15''.61 + v_3 \\ NSO &= \ \ 23^\circ\ 08'\ 05''.26 + v_4 \\ SON &= \ \ 34^\circ\ 40'\ 39''.66 + v_7 \\ \hline \text{Sum} &= 180^\circ\ 00'\ 00''.53 + v_3 + v_4 + v_7 \\ \text{Theoretical sum} &= 180^\circ\ 00'\ 00''.05 = 180^\circ + \varepsilon \end{aligned}$$

and the angle equation is formed by equating these sums. The result is

$$v_3 + v_4 + v_7 + 0''.48 = 0$$

Similarly from the triangle Lester, S. Base, Oneota, the angle equation is

$$v_6 + v_7 + v_8 + v_9 + 1''.10 = 0$$

125. *Number of Angle Equations in a Net.*—It is to be expected that in a triangulation net some of the lines will be sighted over in both directions, and some in only one direction. If these latter lines are omitted the number of angle equations will remain unaltered. Thus in our Lake Superior quadrilateral (Fig. 19) the line *NL* has been

* We confine ourselves throughout to triangles to which Legendre's theorem is applicable. For very large triangles other formulas for spherical excess must be used. See, for example, Helmert, *Theorieen d. höheren Geodäsie*, vol. i. p. 362.

sighted over from N, but not from L, so that we have only two angle equations: namely, those resulting from the triangles ONS, OLS, just as if the figure had been of the form of Fig. 21, in which the line NL is omitted.

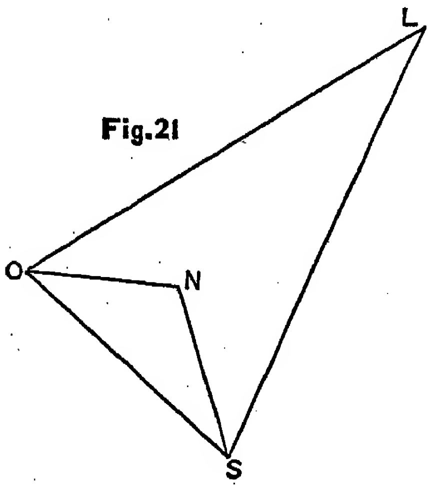


Fig. 21

Generally, if s is the number of stations occupied, the polygon forming the outline of the net will give rise to one angle equation. Each diagonal that is drawn will form a figure, giving rise to an additional angle equation. Hence if in the net there are l_1 lines sighted over in both directions, the number of diagonals will be $l_1 - s$, and the number of angle equations

$$l_1 - s + 1$$

If l_2 of the lines are sighted over in one direction only, and l is the total number of lines in the figure, then since $l_1 = l - l_2$, the number of angle equations would be expressed by

$$l - l_2 - s + 1$$

Thus in the figure the polygon $LONS$ gives an angle equation, and the line OS gives rise to a second angle equation from either the triangle ONS or OLS. We might, therefore, form the equations from either of the three sets of figures,

$LONS$	$LONS$	ONS
ONS	OLS	OLS

and should have respectively

$$v_1 + v_2 + v_5 + v_8 + v_9 + 3.06 = 0$$
$$v_3 + v_4 + v_7 + 0.48 = 0$$

$$v_1 + v_2 + v_5 + v_8 + v_9 + 3.06 = 0$$
$$v_6 + v_7 + v_8 + v_9 + 1.10 = 0$$

$$v_3 + v_4 + v_7 + 0.48 = 0$$
$$v_6 + v_7 + v_8 + v_9 + 1.10 = 0$$

which pairs of equations, by means of the relations already found (pp. 258, 261),

$$v_1 + v_2 + v_3 + 1.37 = 0$$
$$v_4 + v_5 - v_6 + 1.07 = 0$$
$$v_3 + v_4 + v_7 + 0.48 = 0$$

reduce to the same two equations for each set of polygons.

Ex. In the quadrilateral $ABCD$, in which all of the 8 angles are measured, show that there are three independent angle equations, and that these equations may be found from the following 8 sets of figures:

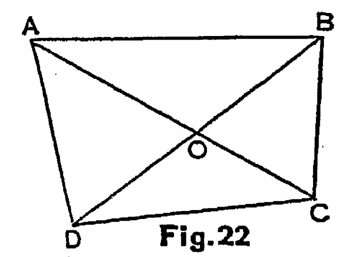


Fig. 22

ABD, ABC, ACD; ABD, ABC, $ABCD$;
ABD, ACD, $ABCD$;
BDA, BCD, BCA; BCA, BCD, $BCDA$;
CDB, CAB, CDA; CDB, CDA, $CDBA$;
DAB, DBC, DAC.

126. *The Side Equations.*—In a single triangle, or in a simple chain of triangles, the length of any assigned side can be computed from a given side in but one way. When the triangles are interlaced this is not the case.

Thus in Fig. 21 any side can be computed from NS in but one way. The only condition equations apart from the local equations would be the two angle equations. But in Fig. 19, in which the line NL is sighted over from N, we have the further condition that the lines OL, NL, SL intersect in the same point, L. The figure plotted from the measured values would be of the form of Fig. 23.

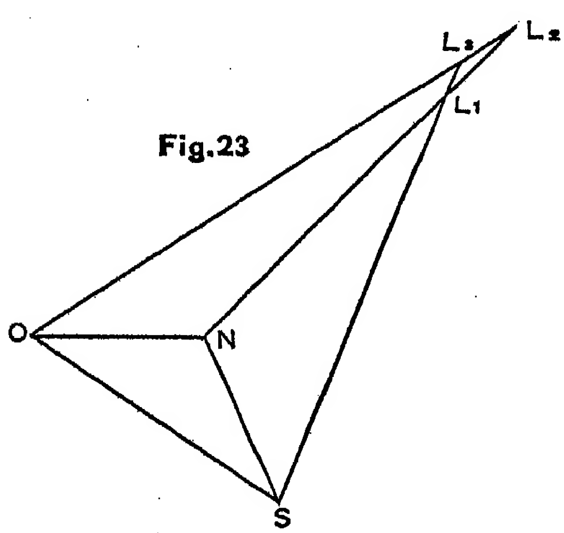


Fig. 23

To express in the form of an equation the condition that the three points L_1, L_2, L_3 must coincide, we proceed as follows: Starting from the base NS, we may compute SL_1 directly

from the triangle SNL_1 and SL_2 from the triangles SON, SOL_2. This gives

$$\frac{\sin SN}{\sin SL_1} = \frac{\sin SL_1N}{\sin SNL_1}$$

$$\frac{\sin SN}{\sin SL_2} = \frac{\sin SON}{\sin SNO}\ \frac{\sin SL_2O}{\sin SOL_2}$$

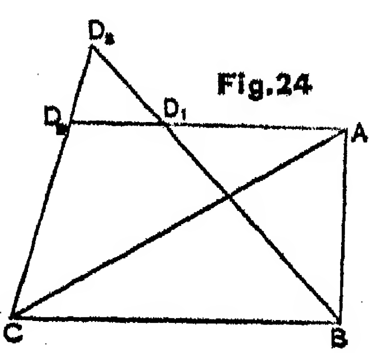
But SL_1 must be equal to SL_2

Hence the condition equation is

$$\frac{\sin SLN}{\sin SNL}\ \frac{\sin SNO}{\sin SON}\ \frac{\sin SOL}{\sin SLO} = 1$$

which is called a *side equation* or *sine equation*.

Ex. In the figure $ABCD_2D_1A$, the three angle equations,



$$\begin{aligned} D_1AB + ABD_1 + BD_1A &= 180^\circ + e_1 \\ ABC + BCA + CAB &= 180^\circ + e_2 \\ BCD_2 + CD_2B + D_2BC &= 180^\circ + e_3 \end{aligned}$$

given by the triangles D_1AB, ABC, BCD_2, may be satisfied and yet the figure not be a perfect quadrilateral. Show by equating the values of BD_1 and BD_2 that the further condition necessary is

$$\frac{\sin DAB\ \sin BCA\ \sin CDB}{\sin BDA\ \sin CAB\ \sin BCD} = 1$$

The side equation

$$\frac{\sin SLN}{\sin SNL}\ \frac{\sin SOL}{\sin SLO}\ \frac{\sin SNO}{\sin SON} = 1$$

gives the identical relation

$$\frac{\sin SN}{\sin SL}\ \frac{\sin SL}{\sin SO}\ \frac{\sin SO}{\sin SN} = 1$$

Hence in forming a side equation we may proceed mechanically in this way. Write down the scheme

$$\frac{SN}{SL}\ \frac{SL}{SO}\ \frac{SO}{SN} = 1$$

the numerator and denominator each being formed by the lines radiating from the point S in order of azimuth, and the first denominator being the second numerator. The side equation results from replacing the sides by the sines of the angles opposite to them.

The point S is called the *pole* of the quadrilateral for this equation.

127. *Position of Pole.*—It is easily seen that in forming the side equation any vertex may be taken as pole. For plotting the figure from the angles of the triangles ONS, OLS, the side equation with pole at S means that the points L_1 and L_2 must coincide. The side equation with pole at N means that L_1, L_3 coincide, and with pole at O that L_2, L_3 coincide. If any one of these conditions is satisfied the others are also satisfied, as each amounts to the same condition that L is not three points but one point.

Similar reasoning will show that by plotting the figure from $LONS$, ONS, the side equations formed by taking the poles at N, L, S, mean that O is not three points but one point, and so on. Hence the side equation formed from any vertex as pole in connection with the angle equations fixes each point of the figure definitely and removes all contradictions from it.

It will be noticed that the reasoning is in no way affected by the line NL being sighted over in only one direction.

Ex. 1. In a quadrilateral $ABCD$, in which all of the 8 angles are measured, show that of the 15 side equations that may be formed, 7 only are different in form, and that by taking the angle equations into account all of them may be reduced to a single form.

Also show that there are 56 ways of expressing the three angle and one side equations necessary to determine the quadrilateral.

Ex. 2. Examine the truth of the following statement. In a quadrilateral an angle equation may be replaced by a side equation, so that the quadrilateral may be determined by 3 angle equations and one side equation, 2 angle equations and 2 side equations, one angle equation and 3 side equations, the number of conditions remaining four, and the four not being all of one kind.

128. If the triangulation net, instead of involving quadrilaterals only, involves central polygons, such that, in computing the lengths of the sides, we can pass from one side to any other through a chain of triangles, the same process is followed in forming the side equations as in a quadrilateral.

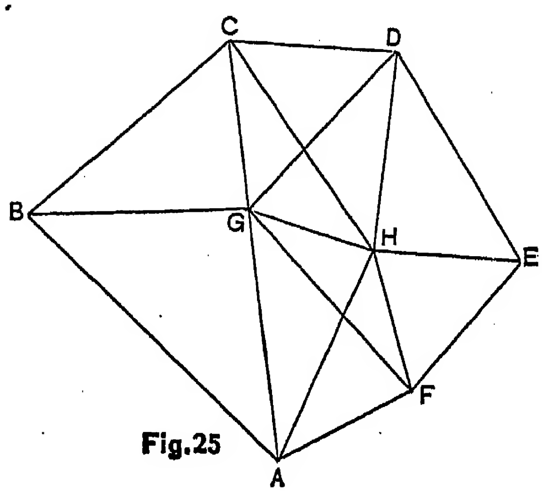


Fig. 25

Thus in the figure which represents part of the triangulation of Lake Erie west of Buffalo Base there are side equations from

the quadrilaterals *CDHG*, *GHFA*
the pentagons *GABCH*, *HGDEF*

The scheme for the pentagonal side equation *GABCH*, for example, would be just as in the case of a central quadrilateral, taking G as pole,

$$\frac{GA}{GB}\ \frac{GB}{GC}\ \frac{GC}{GH}\ \frac{GH}{GA} = 1$$

and the side equation

$$\frac{\sin GBA}{\sin GAB}\ \frac{\sin GCB}{\sin GBC}\ \frac{\sin GHC}{\sin GCH}\ \frac{\sin GAH}{\sin GHA} = 1$$

129. *Reduction to the Linear Form.*—Thus far we have considered the side equations in their rigorous form. But in order to carry through the solution by combining them with the other condition equations they must be reduced to the linear form. We proceed to show how this may be done. (See Art. 7.)

Let the side equation be

$$\frac{\sin V_1}{\sin V_2}\ \frac{\sin V_3}{\sin V_4}\ \cdots = 1 \qquad (1)$$

where V_1, V_2, . . . denote the most probable values of the angles. Let M_1, M_2, . . . denote the measured values, and

$v_1, v_2, \ldots$ the most probable corrections to these values; then the equation may be written

$$\frac{\sin(M_1 + v_1)}{\sin(M_2 + v_2)} \; \frac{\sin(M_3 + v_3)}{\sin(M_4 + v_4)} \ldots = 1 \qquad (2)$$

Taking the log. of each side of this equation and expanding by Taylor's theorem, we have, retaining the first powers of the corrections only,

$$\log \sin M_1 + \frac{d}{dM_1}(\log \sin M_1)\, v_1$$
$$- \left\{ \log \sin M_2 + \frac{d}{dM_2}(\log \sin M_2)\, v_2 \right\} + \ldots = 0 \quad (3)$$

which may be written in two forms for computation:

First, if the corrections to the angles are expressed in seconds, we may put

$$\frac{d}{dM_1}(\log \sin M_1) = \delta'$$

where δ' is the tabular difference for $1''$ for the angle M_1 in a table of log. sines. Then we have

$$\delta' v_1 - \delta'' v_2 + \ldots + \log \sin M_1 - \log \sin M_2 + \ldots = 0$$

that is,

$$[\delta v] = l \qquad (4)$$

where l is a known quantity.

Secondly, we may replace

$$\frac{d}{dM_1}(\log \sin M_1) \quad \text{by} \quad \text{mod} \sin 1'' \cot M_1$$

where mod denotes the modulus of the common system of logarithms. Eq. 3 may then be arranged

$$\text{Cot}\, M_1\, v_1 - \cot M_2\, v_2 + \ldots$$
$$= \frac{1}{10^7 \text{ mod} \sin 1''}(\log \sin M_2 - \log \sin M_1 + \ldots) \quad (5)$$

if the seventh place of decimals is chosen as the unit.

35

For convenience of computation there is not much to choose between the two forms. The second is perhaps, on the whole, to be preferred in ordinary triangulation work with well-shaped triangles.

For the method of computing log. sines and log. differences for small angles or for angles near 180°, and also if a ten-place table is used, see Art. 7.

130. *Check Computation.*—The side equation deduced from spherical triangles must also follow from the corresponding plane triangles, the angles of each spherical triangle being transformed according to Legendre's theorem; that is, for example, we should obtain the same constant term l by reducing to the linear form the equation

$$\frac{\sin SLN}{\sin SNL}\,\frac{\sin SOL}{\sin SLO}\,\frac{\sin SNO}{\sin SON}=1$$

or the equation

$$\frac{\sin\left(SLN-\frac{\varepsilon_1}{3}\right)}{\sin\left(SNL-\frac{\varepsilon_1}{3}\right)}\,\frac{\sin\left(SOL-\frac{\varepsilon_2}{3}\right)}{\sin\left(SLO-\frac{\varepsilon_2}{3}\right)}\,\frac{\sin\left(SNO-\frac{\varepsilon_3}{3}\right)}{\sin\left(SON-\frac{\varepsilon_3}{3}\right)}=1$$

where ε_1, ε_2, ε_3 denote the spherical excesses of the triangles SNL, SOL, and SON respectively.

It affords a check of the accuracy of the numerical work to compute the side equation with both the spherical angles and the plane angles. It is evidently simpler to use the spherical angles, so that if but a single computation is to be made they should be chosen.

For a check of the coefficients of the corrections we have, by expanding the second equation by Taylor's theorem, the relations

$$\varepsilon_1(\delta'-\delta'')+\varepsilon_2(\delta'''-\delta'''')+\varepsilon_3(\delta'''''-\delta'''''')=0$$

or

$$\varepsilon_1(\cot SLN-\cot SNL)+\varepsilon_2(\cot SOL-\cot SLO)$$
$$+\varepsilon_3(\cot SNO-\cot SON)=0$$

for the first and second forms of reduction respectively.

This useful check is given by Andræ.

Ex. The quadrilateral N. Base, S. Base, Oneota, Lester (Fig. 19).
Take the pole at Lester.
We have the scheme

$$\frac{LS}{LN}\,\frac{LN}{LO}\,\frac{LO}{LS} = 1$$

from which we write down the side equation

$$\frac{\sin LNS \sin LON \sin LSO}{\sin LSN \sin LNO \sin LOS} = 1$$

that is,

$$\frac{\sin(M_2 + v_2)\sin(M_8 + v_8)\qquad\sin(M_6 + v_6)}{\sin(M_5 + v_5)\sin(M_1 + v_1)\sin(M_7 + M_8 + v_7 + v_8)} = 1$$

First Form of Reduction.

$$\begin{aligned}
\log\sin(113°\ 39'\ 05''.07 + v_2) &= 9.9618969{,}7 - 9{,}22\,v_2\\
\log\sin(\ 43°\ 46'\ 26''.40 + v_8) &= 9.8399903{,}4 + 21{,}98\,v_8\\
\log\sin(\ 70°\ 39'\ 24''.69 + v_6) &= 9.9747656{,}9 + 7{,}39\,v_6\\
&530{,}0\\
\log\sin(\ 47°\ 31'\ 20''.41 + v_5) &= 9.8677859{,}5 + 19{,}28\,v_5\\
\log\sin(124°\ 09'\ 40''.69 + v_1) &= 9.9177470{,}2 - 14{,}29\,v_1\\
\log\sin(\ 78°\ 27'\ 06''.06 + v_7 + v_8) &= 9.9911180{,}3 + 4{,}30(v_7 + v_8)\\
&510{,}0
\end{aligned}$$

Hence the side equation in the linear form is

$$14.29\,v_1 - 9.22\,v_2 - 19.28\,v_5 + 7.39\,v_6 - 4.30\,v_7 + 17.08\,v_8 + 20.0 = 0$$

the unit being the seventh place of decimals.

Check of the constant term by computing the log. sines after deducting from each angle ⅓ of the spherical excess of the triangle to which it belongs.

Angle.	*Log Sin.*	*Angle.*	*Log Sin.*
113° 39' 05".00	9.9618970,3	47° 31' 20".34	9.8677858,2
43° 46' 26".36	9.8399902,5	124° 09' 40".65	9.9177470,7
70° 39' 24".48	9.9747656,0	78° 27' 05".93	9.9911179,8
	528,8		508,7

+ 20,1

agreeing closely with the value found from the spherical angles.

Check of the coefficients.

$$0.19(-9.22 - 19.28) + 0.12(21.98 + 14.29) + 0.37(7.39 - 4.30) = +0.08$$

This check would have been closer had the spherical excesses been carried out to three places of decimals. We have taken 0".19, 0".05, and 0".12 for the

excesses of the single triangles *LNS*, *ONS*, *LNO*, and 0".37 for the sum triangle *LSO*. The first two are more nearly 0".105 and 0".055.

Second Form of Reduction.

Log Sin.		*Log Sin.*
$9.9618969,7 - 0.4380 v_2$		$9.8677859,5 + 0.9156 v_4$
$9.8399903,4 + 1.0437 v_8$		$9.9177470,2 - 0.6786 v_1$
$9.9747656,9 + 0.3510 v_6$		$9.9911180,3 + 0.2043(v_7 + v_8)$
530,0		
510,0		
20,0	log 1.30103	
$\dfrac{1}{10^7 \text{ mod } \sin 1''}$	log 8.67664	
	9.97767	0.950

and the side equation is

$$0.6786 v_1 - 0.4380 v_2 - 0.9156 v_4 + 0.3510 v_6 - 0.2043 v_7 + 0.8394 v_8 + 0.950 = 0$$

This result may be checked in the same way as in the first form.

In reducing a side equation to the linear form the coefficients of the corrections should be carried out to one place of decimals farther than the absolute term. This for a short computation would be unnecessary, but in the reduction of an extensive triangulation net it is rendered necessary by the accumulation of errors from the dropping of the last figures in products and quotients.

It will be noticed that in the preceding example we have carried out the log. sines to 8 places of decimals, the seventh place being the unit. This is amply sufficient, in primary work, for our present methods of measurement, as already indicated in Art. 34. Indeed, it is in general sufficient to carry them to 7 places of decimals only, an ordinary 7-place table being used. As there is no 8-place table published, the labor of forming the log. sines with a 10-place table, and then cutting down the results to 8 places, is, in very many cases, hardly justified by the extra precision attained.

For a secondary triangulation 6 places of decimals, and for a tertiary triangulation 5 places, may be used.*

* On the Coast Survey and Lake Survey the practice in primary triangulation has been to carry out the log. sines to 10 places of decimals. On the English Ordnance Survey they were also carried out to 10 places, but on the more modern Great Trigonometrical Survey of India to 7 places only. In the triangulation of Denmark, Andræ used 8 places; and in this he has been followed by the Prussian, Italian, and other modern European surveys.

131. We have seen that the coefficients of the corrections in a side equation are given by the differences for 1" of the log. sines of the angles, or by the cotangents of the angles that enter. Now, since an equable distribution of errors arising from the approximate computation is best attained by securing the greatest possible equality of coefficients throughout the condition equations, and since the coefficients of the corrections in the angle equations are $+1$ or -1, it follows that it would be most convenient to put the side equations on the same footing as the angle equations. To do this we may divide the side equation by such a number as will make the average value of the coefficients equal to unity. This, for angles ordinarily met with in triangulation, would be effected by taking the sixth place of decimals as the unit in the side equation. Thus in our example, dividing by 10, which is approximately the mean of the coefficients, and which amounts to the same thing as expressing the log. differences in units of the sixth place of decimals, the equation may be written

$$1.43v_1 - 0.92v_3 - 1.93v_4 + 0.74v_5 - 0.43v_6 + 1.77v_8 + 2.00 = 0$$

It would have been equally correct to have multiplied each of the angle equations by 10, and so have put them on the same footing as the side equations. Dividing the side equations is, however, to be preferred, as the coefficients are made smaller throughout, and the formation and solution of the normal equations is consequently easier.

A striking difference between condition equations and observation equations is here brought out. As a condition equation expresses a rigorous relation among the observed quantities altogether independent of observation, it may be multiplied or divided by any number without affecting that relation; with an observation equation, on the other hand, the effect would be to increase or diminish its weight. (Compare Art. 58.)

132. *Position of Pole.*—In a quadrilateral, taking any of the vertices as pole, the conclusion was reached in Art. 127 that

any one of the resulting forms of side equation was as good as any other in satisfying the conditions imposed. But when a side equation is reduced to the linear form and is no longer rigorous the question deserves farther notice.

Two points are to be considered—precision of results and ease of computation. As regards the first, since the differences in a table of log. sines are more sharply defined for small angles, and these differences are the coefficients of the unknowns in the side equation, it follows that in general that vertex should be chosen which allows the introduction of the acutest angles into the side equation.

Labor of computation will be saved by choosing the pole so that as few sine terms as possible enter. Thus by choosing the pole at O, the intersection of the diagonals (Fig. 22), the side equation would contain 8 terms, whereas if taken at any of the vertices only 6 terms would enter. Also, other things being equal, we should choose that pole which introduces the smallest number of unknowns into the equation, for then the normal equations would be more easily formed.

If the approximate form of solution in Art. 115 is employed it is advantageous to choose the pole at the intersection, O, of the diagonals, as will be seen in the sequel.

133. *Number of Side Equations in a Net.*—A line being taken as a base, its extremities are known. To fix a third point we must know the other two sides of the triangle of which this point is to be the vertex. Hence if we have a net of triangles connecting s stations, two of the stations being the ends of the base, we must have, in order to plot the figure, $2(s-2)$ lines besides the base; that is, $2s-3$ lines in all.

Starting from the base, each line in this figure can be computed in but one way, but any additional line, whether observed over in one or both directions, can be computed in two ways, and therefore gives rise to a side equation. If, then, the total number of lines in the figure is l, the num-

ber of side equations, as indicated by the number of superfluous lines, is

$$l - 2s + 3.$$

134. *Check of the Total Number of Conditions.*—Leaving local equations out of account, if l is the number of lines in a figure sighted over in both directions, and s the number of stations, the total number of angles in the figure is $2l - s$. If l_2 of these lines have been sighted over in one direction only, the number of angles is reduced to $2l - l_2 - s$.

Now, the number of angle equations in the figure is

$$l - l_2 - s + 1$$

and the number of side equations is

$$l - 2s + 3$$

$\therefore$ the total number of condition equations is

$$2l - l_2 - 3s + 4, \quad \text{that is, } n - 2s + 4$$

where n is the number of angles in the figure.

But we have seen in Art. 123 that the total number of conditions to be satisfied among the same n angles is

$$n - 2s + 4$$

We conclude, therefore, that the conditions are completely covered by the angle and side equations.

135. **Manner of Selecting the Angle and Side Equations.**—In the selection of the angle and side equations in a triangulation net we have two dangers to guard against: first, that we omit no necessary conditions, and, second, that we admit no unnecessary ones. The rule usually followed is to start from some line as base, and plot, the figure proceeding from station to station, writing down the conditions that express the connections of each station to the net as the net grows.

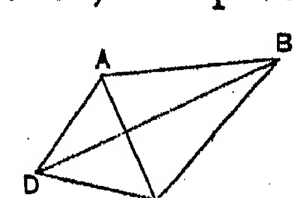


Fig. 26

For example, let Fig. 27 represent a triangulation net. Taking AB as base (Fig. 26), a third station, C, gives an angle equation from the triangle ABC. A

fourth station, D, gives in addition angle equations from ABD, ACD; side equation from $ABCD$. A fifth point, E, gives in addition (Fig. 27) angle equations from ABE, ADE; side equation from $ABDE$.

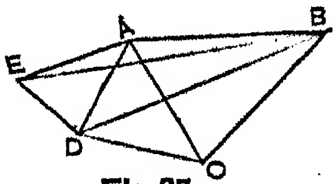


Fig. 27

We have thus in the complete figure (Fig. 27) 5 angle equations and 2 side equations.

Check. Number of stations $= 5$
Number of lines $= 9$
$\therefore$ Number of angle equations $= 9 - 5 + 1 = 5$
and Number of side equations $= 9 - 2 \times 5 + 3 = 2$

As an illustration of the difficulties which may arise in selecting the angle and side equations, let us take the triangulation around the Chicago Base (1877). It is represented in the figure.

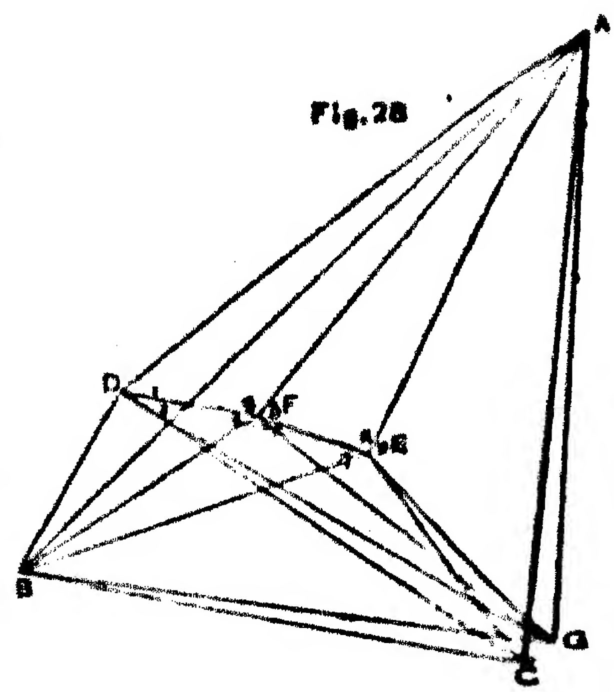


Fig. 28

From the rules laid down in Arts. 125, 133 it follows that there are in the adjustment 10 angle equations and 8 side equations.

The peculiarity of the system is that the station F is very close to the base line DE. Thus the angle equation from the triangle DEF

$$\begin{aligned}
EDF &= 00^\circ\ 00'\ 00''.815 + v_1 \\
FED &= 00^\circ\ 00'\ \ 1''.185 + v_2 \\
DFE &= 179^\circ\ 59'\ 56''.733 + v_3 + v_4 \\
\hline
& 179^\circ\ 59'\ 58''.733 + v_1 + v_2 + v_3 + v_4 \\
& 180^\circ\ 00'\ 00''.000 \\
\hline
\therefore \quad 0 &= -1''.267 + v_1 + v_2 + v_3 + v_4
\end{aligned}$$

In the selection of the side equations it is advisable to avoid those quadrilaterals in the figure which are entangled

with the above triangle; that is, the quadrilaterals *AEDF*, *BDFE*, *GDFE*. For example, if we take the quadrilateral *GDFE* we have, in units of the seventh place of decimals, pole at *G*, the side equation

$$54.7721v_1 - 0.0008v_2 + 0.0015v_3 - 40.7066(v_5 + v_6) \\ + 26.1803v_7 - 0.0003(v_8 + v_9) = 40.13$$

Now, since the coefficients of v_2, v_3, v_8, v_9 are each less than 2 in the third place, this equation is nearly the same as

$$54.772v_1 - 40.707(v_5 + v_6) + 26.180v_7 = 40.13$$

or dividing by 40 and replacing $v_5 + v_6$ by $-v_3 - v_4$,

$$1.369v_1 + 1.018(v_3 + v_4) + 0.654v_7 = 1$$

which is nearly the same as the angle equation from the triangle *DEF*.

Similarly the quadrilaterals *AEDF*, *BDEF* give respectively, neglecting coefficients less than 5 in the fifth place,

$$1.219v_1 + 0.208v_7 + 0.722(v_3 + v_4) = 0.790$$
$$0.256v_1 + 2.466v_7 + 1.343(v_3 + v_4) = 0.836$$

both of which express approximately the same relations among the angles of the small triangle *DEF* as the angle equation formed from this triangle. We therefore conclude that in the formation of the condition equations other quadrilaterals than these should be chosen.

136. Again, in selecting the side equations in a net care must be taken that only independent conditions are chosen. Thus in Fig. 28 we might have chosen the following eight quadrilaterals from which to form the side equations:

AGBD, *AGDE*, *AGBE*, *BDFG*,
BFEG, *AGDF*, *ACBE*, *BDEG*.

A careful examination will show that these figures are not independent. For, taking the four, *AGBD*, *AGDE*, *AGBE*,

BDEG, and choosing *G* as pole, which is common to all, we have the side equations

$$\frac{\sin ADG}{\sin DAG}\,\frac{\sin DBG}{\sin BDG}\,\frac{\sin BAG}{\sin ABG} = 1$$

$$\frac{\sin DAG}{\sin ADG}\,\frac{\sin EDG}{\sin DEG}\,\frac{\sin AEG}{\sin EAG} = 1$$

$$\frac{\sin ABG}{\sin BAG}\,\frac{\sin BEG}{\sin EBG}\,\frac{\sin EAG}{\sin AEG} = 1$$

$$\frac{\sin BDG}{\sin DBG}\,\frac{\sin DEG}{\sin EDG}\,\frac{\sin EBG}{\sin BEG} = 1$$

which equations multiplied together reduce to the identical form

$$1 = 1$$

showing that from any three the fourth may be found.

The entrance of mutually dependent conditions would, however, be detected in the course of the solution of the normal equations, as we should arrive at two identical equations; or, in other words, one of the correlates would become indeterminate.

If the rule given on p. 273 is followed closely this repetition of conditions will hardly occur.

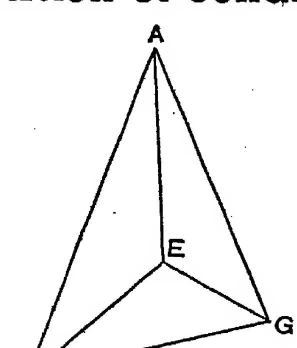


Fig. 29

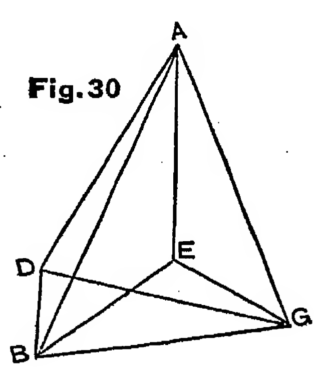


Fig. 30

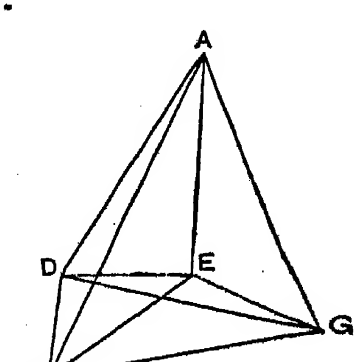


Fig. 31

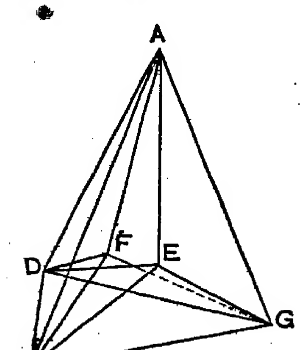


Fig. 32

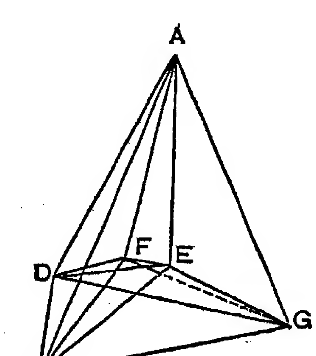


Fig. 33

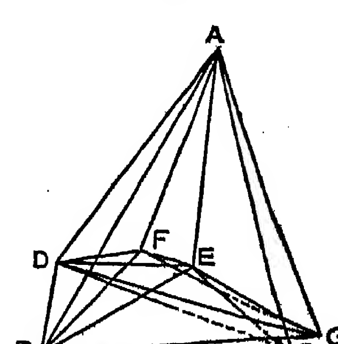


Fig. 34

137. In the example chosen (Fig. 28) the selection of the angle and side equations may be made in the following order, proceeding from the line AB, and adding station to station and line to line till the complete figure is reached:

	Angle equations.	*Side equations.*
From Fig. 29,	Triang. AGB	
	" AEB	
	" BEG	
		Quad. $AGBE$
From Fig. 30 in addition,	" DAG	
	" BDG	
		" $AGBD$
From Fig. 31 in addition,	" DAE	
		" $AGDE$
From Fig. 32 in addition,	" DAF	
		" $AGDF$
	" BDF	
		" $BDFG$
From Fig. 33 in addition,	" BFE	
		" $BFEG$
From Fig. 34 in addition,	" BAC	
		" $ACBE$
		" $CBDE$

In all 10 angle equations and 8 side equations, as should be (p. 274).

138. We finally notice the *arrangement* for solution of the condition equations in the net adjustment. On first thoughts it might seem that it would be well to arrange the angle equations and side equations in two separate sets, and so carry them forward for solution. This was done in some of the older work.* The only objection to doing it is that the process of finding the corrections is more troublesome. Ex-

* See for an example *Verification and Extension of La Caille's Arc of Meridian*, by Sir Thomas Maclear, vol. i. pp. 496 seq.

perience shows that the solution of a series of normal equations is much facilitated if the coefficients are arranged as the steps of a stair rather than irregularly. Thus

$$
\begin{aligned}
[aa]x + [ab]y &= [al] \\
[ab]x + [bb]y + [bc]z &= [bl] \\
[bc]y + [cc]z &= [cl]
\end{aligned}
$$

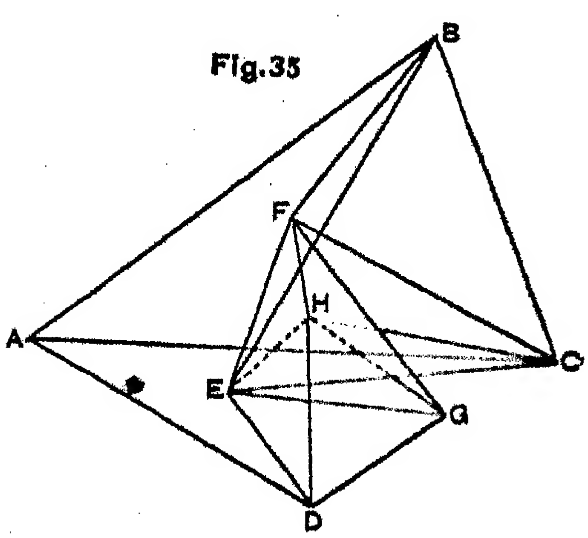
is a more convenient form for solution than

$$
\begin{aligned}
[aa]x + [ab]y &= [al] \\
[cc]z + [bc]y &= [cl] \\
[ab]x + [bc]z + [bb]y &= [bl]
\end{aligned}
$$

The condition equations should therefore be so arranged that, as far as possible (and it cannot always be done at the first trial), the normal equations will fall in the first of the above forms rather than the second. A good rule is to begin with an angle equation, proceeding from triangle to triangle until the points gone over are covered by a side equation, and then introduce it. Continue the process with the remaining sets of triangles.

As to the angle equations themselves, it matters but little which triangles are taken to form them. It is better, however, to avoid triangles with very small angles, such as DEF in Fig. 28, and, where angles so very small occur, to avoid triangles involving angles immediately contiguous to these small angles.

Fig. 35



139. In the explanation of the formation of the side equations we have assumed that, the computation of one side from another assumed as base, by different routes we proceed through chains of triangles. The omission of certain lines in the measurement may make it necessary to proceed through polygons, and then the formation of the side equations becomes more complicated.

A good illustration occurs in the triangulation of Lake Superior* (1871) near Keweenaw Base, as shown in Fig. 35.

Here there are 18 lines and 8 stations, requiring 5 side equations. Proceeding from the line DG and adding point to point, we form the first four side equations from the quadrilaterals

$$HGDE, \; HGEF, \; HEFC, \; CEFB$$

With these there is no difficulty.

The fifth side equation is furnished by the pentagon $ABCED$. We cannot compute directly any specified side of this pentagon from another side through triangles only. A little artifice, however, will enable us to do this. Suppose the line CD drawn. Call the angle $GDC = x$ and $ECD = y$. This new line CD gives an additional side equation. We take the two pentagons $ABCED$, $DEGFC$.

From the first pentagon, $ABCED$,

$$\frac{CE}{CD}\,\frac{CD}{CA}\,\frac{CA}{CB}\,\frac{CB}{CE} = 1$$

or

$$\frac{\sin (EDG + x)}{\sin CED}\;\frac{\sin CAD}{\sin (ADG + x)}\;\frac{\sin CBA}{\sin CAB}\;\frac{\sin CEB}{\sin CBE} = 1 \qquad (1)$$

and from the second pentagon, $DEGFC$, pole at E,

$$\frac{ED}{EG}\,\frac{EG}{EF}\,\frac{EF}{EC}\,\frac{EC}{ED} = 1$$

or

$$\frac{\sin EGD}{\sin EDG}\;\frac{\sin EFG}{\sin EGF}\;\frac{\sin ECF}{\sin EFC}\;\frac{\sin (EDG + x)}{\sin y} = 1 \qquad (2)$$

Now, y can be expressed in terms of x from the triangle CDE; thus

$$y + CED + EDG + x = 180 + \epsilon \qquad (3)$$

where ϵ is the spherical excess of the triangle CDE.

* *Report of Chief of Engineers*, 1872.

Eliminating x from equations 1, 2, 3, and the required side equation results.

To find it write (1) in the form

$$\cot(EDG+x) = \frac{1}{\sin ADE}\,\frac{\sin CAD}{\sin CED}\,\frac{\sin ABC}{\sin BAC}\,\frac{\sin BEC}{\sin CBE} - \cot ADE$$

from which the value of x follows at once.

Hence since y is known from (3), all the angles in (2) are known, and the solution is finished in the usual way.

If chains of triangles intersect so as to form a closed polygon, this method may be employed in simple cases. Such forms are, however, in general better treated by other methods, which will be found in works on the higher geodesy.

140. *Ex.*—**Adjustment of the Quadrilateral NSOL** (Fig. 19).

The method of forming the condition equations having now been explained, we are ready to adjust the quadrilateral $NSOL$, as promised in Art. 120.

The condition equations have all been formed in the preceding sections. Collecting them, we have:

Local equations (Ex. 1, 2, Art. 121)

$$v_1 + v_2 + v_3 = -1.37$$
$$v_4 + v_5 - v_6 = -1.07$$

Angle equations (Ex. Art. 124)

$$v_3 + v_4 + v_7 = -0.48$$
$$v_6 + v_7 + v_8 + v_9 = -1.10$$

Side equation, the unit being the sixth place of decimals (Ex. Art. 130),

$$1.43v_1 - 0.92v_2 - 1.93v_5 + 0.74v_6 - 0.43v_7 + 1.77v_8 = -2.00$$

The methods of solution have been explained in Chap. V., and we shall proceed in the order there given for the four forms.

FIRST SOLUTION—METHOD OF INDEPENDENT UNKNOWNS.

There being 9 unknowns and 5 condition equations connecting them, there must be 4 independent unknowns. We shall choose v_1, v_2, v_4, v_5. Expressing all of the unknowns in terms of these four, we write the equations in the form of observation equations, as follows (see Art. 109):

$$
\begin{array}{llllllll}
v_1 = + & v_1 & & & & & \text{weight} & 2 \\
v_2 = & & + \; v_2 & & & & \text{"} & 2 \\
v_3 = - & v_1 - & v_2 & & & - 1.37 & \text{"} & 14 \\
v_4 = & & & + \; v_4 & & & \text{"} & 23 \\
v_5 = & & & & + \; v_5 & & \text{"} & 6 \\
v_6 = & & & + \; v_4 + & v_5 & + 1.07 & \text{"} & 7 \\
v_7 = + & v_1 + & v_2 - & v_4 & & + 0.89 & \text{"} & 31 \\
v_8 = - & 0.565 v_1 + & 0.763 v_2 - & 0.661 v_4 + & 0.672 v_5 & - 1.361 & \text{"} & 1 \\
v_9 = - & 0.435 v_1 - & 1.763 v_2 + & 0.661 v_4 - & 1.672 v_5 & - 1.699 & \text{"} & 8
\end{array} \qquad (1)
$$

Hence the normal equations

v_1	v_2	v_4	v_5	Const.	
+ 48.83	+ 50.70	− 32.93	+ 5.44	− 53.45	
	+ 72.45	− 40.83	+ 24.09	− 69.70	
		+ 64.93	− 2.29	+ 28.18	
			+ 35.82	− 29.30	
				+ 83.79	= $[pll]$

Solving these equations (page 284), we have the values of the corrections

$$v_1 = -0''.82 \qquad v_4 = -0''.22$$
$$v_2 = -0''.36 \qquad v_5 = -0''.47$$

and thence from the condition equations

$$v_3 = -0''.19 \qquad v_8 = -1''.33$$
$$v_6 = +0''.38 \qquad v_9 = -0''.08$$
$$v_7 = -0''.07$$

These corrections applied to the measured values of the angles give the most probable values as follows:

$$
\begin{array}{llllllll}
M_1 = 124^\circ & 09' & 39''.87 & \qquad & M_6 = 70^\circ & 39' & 24''.98 \\
M_2 = 113^\circ & 39' & 04''.71 & & M_7 = 34^\circ & 40' & 39''.59 \\
M_3 = 122^\circ & 11' & 15''.42 & & M_8 = 43^\circ & 46' & 25''.07 \\
M_4 = 23^\circ & 08' & 05''.04 & & M_9 = 30^\circ & 53' & 30''.73 \\
M_5 = 47^\circ & 31' & 19''.94 & & & &
\end{array}
$$

The Precision of the Adjusted Values.—(a) To find the m. s. e. of an observation of the unit of weight (Arts. 99, 101).

From the above values of the residuals v

$$[pvv] = 7.53$$

Check of $[pvv]$. Carrying through the solution of the normal equations the extra column required by the sum $[pll]$, we find (page 284)

$$[pvv] = 7.54$$

Hence

$$\mu = \sqrt{\frac{7.54}{9-4}}$$
$$= \pm 1''.23$$

(b) To find the weight and m. s. e. of the adjusted value of an angle.

Take the angle NLS. Proceeding as in Art. 101, we have

$$F = NLS$$
$$= 180 + \varepsilon - (M_2 + v_2 + M_5 + v_5)$$
$$\therefore dF = -v_2 - v_5$$

Hence from the extra column, the sixth, carried through the solution of the normal equations (page 284),

$$u_F = 0.053$$

and therefore

$$\mu_F = 1.23\sqrt{0.053}$$
$$= 0''.28$$

(c) To find the weight and m. s. e. of the adjusted value of a side, the base, NS, being supposed to be free from error.

Let us take the side OL. We have

$$F = OL$$

$$= NS\frac{\sin ONS}{\sin SON}\frac{\sin LSO}{\sin OLS}$$

$$= NS\frac{\sin (M_3 + v_3)}{\sin (M_7 + v_7)}\frac{\sin (M_6 + v_6)}{\sin (M_9 + v_9)}$$

For check we shall proceed in two ways.

(1) Expand F directly; then

$$dF = \left(\frac{\partial F}{\partial M_2}v_2 + \frac{\partial F}{\partial M_6}v_6 + \frac{\partial F}{\partial M_7}v_7 + \frac{\partial F}{\partial M_9}v_9\right)\sin 1''$$

$$= -0.0505v_2 + 0.0282v_6 - 0.1160v_7 - 0.1342v_9$$

$$= -0.007v_1 + 0.171v_3 + 0.056v_4 + 0.253v_6$$

by substituting for v_2, v_6, v_7, v_9 their values from equations 1.

Carry through the solution of the normal equations the extra column required by these coefficients, and (see page 284)

$$n_F = 0.0019$$

Hence

$$\mu_F = 1.23\sqrt{0.0019}$$
$$= 0^m.05$$

(2) Take logs. of both members of the equation; then

$$\log F = \log NS + \log\sin(M_2 + v_2) + \log\sin(M_6 + v_6) - \log\sin(M_7 + v_7) - \log\sin(M_9 + v_9)$$

But since NS is constant, we have, in units of the sixth place of decimals,

$$d\log F = -1.33v_2 + 0.74v_6 - 3.04v_7 - 3.52v_9$$
$$= -0.18v_1 + 4.50v_3 + 1.45v_4 + 6.63v_6 \text{ from equations 1.}$$

Hence from the last column added to the solution of the normal equations,

$$n_{\log F} = 1.50 \text{ in units of the sixth place of decimals.}$$

Also,

$$\mu_{\log F} = 1.23\sqrt{1.50}$$
$$= 1.5 \text{ in units of the sixth place of decimals.}$$

Now, since (p. 23)

$$d\log F = \frac{dF}{F}\,\text{mod}$$

and (p. 255)

$$F = 16556m,$$
$$\therefore\ \mu_F = 0^m.06.$$

The solution of the normal equations, with the extra columns required by the weight determinations, is as follows:

v_1	v_2	v_4	v_5	l	f (angle).	f (side).	f (side).
+ 48.83	+ 50.70	− 32.93	+ 5.44	− 53.45		− 0.007	− 0.18
	+ 72.45	− 40.83	+ 24.09	− 69.70	− 1.	+ 0.171	+ 4.50
		+ 64.93	− 2.29	+ 28.18		+ 0.056	+ 1.45
			+ 35.82	− 29.30	− 1.	+ 0.253	+ 6.63
			$[pll]$ =	+ 83.79			
+ 1.	+ 1.0383	− 0.6743	+ 0.1114	− 1.0946		− 0.0001	− 0.0037
	+ 19.8082	− 6.6430	+ 18.4420	+ 14.2038	− 1.	+ 0.1761	+ 4.6871
		+ 42.7253	+ 1.3784	+ 7.8652		+ 0.0527	+ 1.3285
			+ 35.2140	+ 23.3454	− 1.	+ 0.2535	+ 6.6501
				+ 25.2836	0.	0.	+ 0.0007
	+ 1.	− 0.3354	+ 0.9310	+ 0.7176	+ 0.0505	+ 0.0089	+ 0.2366
		+ 40.4972	+ 7.5630	+ 12.6322	− 0.3355	+ 0.1118	+ 2.9002
			+ 18.0445	+ 10.1114	− 0.0687	+ 0.0894	+ 2.2867
				+ 15.0910	+ 0.0505	+ 0.0016	+ 1.1089
		+ 1.	+ 0.1868	+ 0.3119	− 0.0083	+ 0.0027	+ 0.0716
			+ 16.6317	+ 7.7525	− 0.0057	+ 0.0690	+ 1.7452
				+ 11.1510	+ 0.0028	+ 0.0003	+ 0.2076
			+ 1.	+ 0.4661	− 0.0004	+ 0.0003	+ 0.1049
			$[pvv]$ =	+ 7.5376	0.	0.	+ 0.1832
					0.0000	0.0000	0.0007
					0.0505	0.0016	1.1089
					0.0028	0.0003	0.2076
					0.0000	0.0000	0.1832
					0.0533	0.0019	1.5004
					$= u_F$	$= u_F$	$= u_F$

The solution has been carried through to four places of decimals, on account of loss of accuracy arising from dropping figures in multiplications. The resulting values of the corrections have been cut down to two places of decimals. The work was done with a machine, as explained on p. 161, the reciprocals of the diagonal terms being used so as to avoid divisions. Thus the first reciprocal is 0.02048.

SECOND SOLUTION—METHOD OF CORRELATES.

Arranging the condition equations in tabular form, we have

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	
weights 2	2	14	23	6	7	31	1	8	
+ 1.43	− 0.92			− 1.93	+ 0.74	− 0.43	+ 1.77		− 2.00
+ 1.	+ 1.	+ 1.							− 1.37
		+ 1.	+ 1.			+ 1.			− 0.48
			+ 1.	+ 1.	− 1.				− 1.07
					+ 1.	+ 1.	+ 1.	+ 1.	− 1.10

The Correlate Equations.

	I.	II.	III.	IV.	V.
$2v_1 =$	+ 1.43	+ 1.			
$2v_2 =$	− 0.92	+ 1.			
$14v_3 =$		+ 1.	+ 1.		
$23v_4 =$			+ 1.	+ 1.	
$6v_5 =$	− 1.93			+ 1.	
$7v_6 =$	+ 0.74			− 1.	+ 1.
$31v_7 =$	− 0.43		+ 1.		+ 1.
$v_8 =$	+ 1.77				+ 1.
$8v_9 =$					+ 1.

The Normal Equations.

I.	II.	III.	IV.	V.	l
+ 5.284	+ 0.255	− 0.014	− 0.427	+ 1.862	− 2.00
	+ 1.071	+ 0.071			− 1.37
		+ 0.147	+ 0.043	+ 0.032	− 0.48
			+ 0.353	− 0.143	− 1.07
				+ 1.300	− 1.10

The solution of these equations gives (see page 287)

$$\begin{aligned} \text{I.} &= -0.3973 \\ \text{II.} &= -1.0749 \\ \text{III.} &= -1.6006 \\ \text{IV.} &= -3.5721 \\ \text{V.} &= -0.6301 \end{aligned}$$

Substituting these values in the correlate equations, the same values of the corrections result as before. Also,

$$[pvv] = 7.53$$

The Precision.—(a) To find the m. s. e. μ of an observation of weight unity. From the values of v we find directly

$$[pvv] = 7.53$$

Checks of $[pvv]$. These are worked out in the solution of the normal equations on page 287, according to the formulas of Art. 111, and give 7.54 and 7.55 respectively.

Hence taking the mean, $[pvv] = 7.54$, and the number of conditions being 5,

$$\mu = \sqrt{\frac{7.54}{5}}$$
$$= 1''.23 \text{ as before.}$$

Compare Ex. 2, Art. 111.

(b) To find the weight and m. s. e. of the adjusted value of an angle.

Take the angle *NLS*.

$$\therefore\ dF = -v_2 - v_5$$

From the values of u, a, b, . . . in the condition equations in connection with the values of f given by this function, we have

$$[uaf] = +0.782 \qquad [udf] = -0.167$$
$$[ubf] = -0.500 \qquad [uef] = 0.$$
$$[ucf] = 0. \qquad [uff] = +0.667$$

Hence from the seventh column in the solution of the normal equations (page 287),

$$u_F = 0.053$$

and

$$\mu_F = 1.23\sqrt{0.053}$$
$$= 0''.28$$

Compare Ex. 4, Art. 111.

(c) To find the weight and mean-square error of the adjusted value of a side, the base being free from error.

Take the side Oneota–Lester.

As in (c), page 282, we have

$$dF = -0.0505v_3 + 0.0282v_6 - 0.1160v_7 - 0.1342v_9$$

Also from the condition equations

$$[uaf] = +0.0046 \qquad [udf] = -0.0040$$
$$[ubf] = -0.0036 \qquad [uef] = -0.0165$$
$$[ucf] = -0.0073 \qquad [uff] = +0.0030$$

Hence from the eighth column in the solution of the normal equations,

$$u_F = 0.0023$$

and finally,

$$\mu_F = 1.23\sqrt{.0023}$$
$$= 0^m.06$$

Solution of the Normal Equations.

I.	II.	III.	IV.	V.	l	f (angle).	f (side).
+ 5.284	+ 0.255	− 0.014	− 0.427	+ 1.862	− 2.00	+ 0.782	+ 0.0046
	+ 1.071	+ 0.071			− 1.37	− 0.500	− 0.0036
		+ 0.147	+ 0.043	+ 0.032	− 0.48		− 0.0073
			+ 0.353	− 0.143	− 1.07	− 0.167	− 0.0040
				+ 1.300	− 1.10		− 0.0165
						+ 0.667	+ 0.0030
+ 1.	+ 0.0483	− 0.0026	− 0.0808	+ 0.3524	− 0.3785	+ 0.1480	+ 0.0009
	+ 1.0587	+ 0.0717	+ 0.0206	− 0.0902	− 1.2731	− 0.5377	− 0.0038
		+ 0.1470	+ 0.0419	+ 0.0369	− 0.4853	+ 0.0021	− 0.0073
			+ 0.3185	+ 0.0075	− 1.2316	− 0.1035	− 0.0036
				+ 0.6438	− 0.3952	− 0.2756	− 0.0182
					+ 0.7570	+ 0.5510.	
	+ 1.	+ 0.0677	+ 0.0204	− 0.0851	− 1.2025	− 0.5078	− 0.0036
		+ 0.1421	+ 0.0404	+ 0.0434	− 0.3991	+ 0.0385	− 0.0070
			+ 0.3181	+ 0.0093	− 1.2068	− 0.0930	− 0.0035
				+ 0.6361	− 0.5037	− 0.3214	− 0.0185
					+ 1.5309	+ 0.2780	+ 0.0030
		+ 1.	+ 0.2843	+ 0.3054	− 2.8086	+ 0.2709	− 0.0493
			+ 0.3066	− 0.0030	− 1.0933	− 0.1039	− 0.0015
				+ 0.6228	− 0.3818	− 0.3332	− 0.0164
					+ 1.1209	+ 0.2676	+ 0.0027
			+ 1.	− 0.0098	− 3.5659	− 0.3389	− 0.0049
				+ 0.6228	− 0.3924	− 0.3342	− 0.0164
					+ 3.8986	+ 0.2324	+ 0.0027
				+ 1.	− 0.6301 = V.	− 0.5366	− 0.0264
					+ 0.2472	+ 0.0531	+ 0.0023
						$= u_F$	$= u_F$

Values of the Unknowns:

I. = − 0.3973
II. = − 1.0749
III. = − 1.6006
IV. = − 3.5721
V. = − 0.6301

I. × l' = − 0.3973 × − 2.00 = 0.79	0.7570
II. × l'' = − 1.0749 × − 1.37 = 1.47	1.5309
III. × l''' = − 1.6006 × − 0.48 = 0.77	1.1209
IV. × l'''' = − 3.5721 × − 1.07 = 3.82	3.8986
V. × l''''' = − 0.6301 × − 1.10 = 0.69	0.2472
7.54	7.5546
= [pvv]	= [pvv]

The values of [pvv] are found from equations 2, 3, Art. 111.

THIRD SOLUTION—SOLUTION IN TWO GROUPS.

The form given in Art. 113 is followed.

The Local Adjustment.

(a) At North Base.

The Observation Equations.

p	(x_1)	(x_2)	l
2	+1		0.
2		+1	0.
14	−1	−1	−1.37

The Normal Equations.

$$\begin{array}{ll} (x_1) & (x_2) \\ 16 + & 14 = -19.18 = [pal] \quad \text{suppose} \\ 14 + & 16 = -19.18 = [pbl] \quad \text{"} \end{array}$$

Solving in general terms.

$$(x_1) = +0.267\,[pal] - 0.233\,[pbl]$$
$$(x_2) = -0.233\,[pal] + 0.267\,[pbl]$$

Hence

$$(x_1) = -0''.64$$
$$(x_2) = -0''.64$$

and

$$(x_3) = +0''.64 + 0''.64 - 1''.37$$
$$= -0''.09$$

Local Angles.

124°	09′	40″.05
113°	39′	04″.43
122°	11′	15″.52

To find the m. s. e. of a single observation.
The value of $[pvv] = [pxx] = 1.75$.
Hence for this station, the number of conditions being $3 - 2$,

$$\mu = \sqrt{\frac{1.75}{3-2}}$$
$$= 1''.3$$

(b) At South Base.

The Observation Equations.

p	(x_4)	(x_5)	l
23	$+1$		0.
6		$+1$	0.
7	$+1$	$+1$	-1.07

The Normal Equations.

$$\begin{array}{ll} (x_4) & (x_5) \\ 30 + & 7 = -7.49 \\ 7 + & 13 = -7.49 \end{array}$$

Hence

$$(x_4) = -0''.13$$
$$(x_5) = -0''.50$$
$$(x_6) = -0''.13 - 0''.50 + 1''.07$$
$$= +0''.44$$

Local Angles.

$$23^\circ \quad 08' \quad 05''.13$$
$$47^\circ \quad 31' \quad 19''.91$$
$$70^\circ \quad 39' \quad 25''.04$$

Also,

$$[pvv] = [pxx] = 3.24$$

$$\therefore \quad \mu = \sqrt{\frac{3.24}{3-2}}$$
$$= 1''.8$$

The General Adjustment.

Most Probable Angles.

At N. Base,	124°	$09'$	$40''.05 + (1)$
	113°	$39'$	$04''.43 + (2)$
	122°	$11'$	$15''.52 - (1) - (2)$
At S. Base,	23°	$08'$	$05''.13 + (4)$
	47°	$31'$	$19''.91 + (5)$
	70°	$39'$	$25''.04 + (4) + (5)$
At Oneota,	34°	$40'$	$39''.66 + (7)$
	43°	$46'$	$26''.40 + (8)$
At Lester,	30°	$53'$	$30''.81 + (9)$

The Angle and Side Equations.

(a) Triangle, N. Base, S. Base, Oneota.

$$
\begin{array}{lll}
\text{Angle } SNO & 122^\circ\ 11'\ 15''.52 & -(1)-(2) \\
\text{“} \quad NSO & 23^\circ\ 08'\ 05''.13 & +(4) \\
\text{“} \quad NOS & 34^\circ\ 40'\ 39''.66 & +(7) \\
\hline
\text{Sum} = & 180^\circ\ 00'\ 00''.31 & \\
180+\varepsilon = & 180^\circ\ 00'\ 00''.05 & \\
\end{array}
$$

$$0 = 0''.26 - (1) - (2) + (4) + (7)$$

(b) Triangle Lester, Oneota, S. Base.

$$
\begin{array}{lll}
\text{Angle } NSO & 70^\circ\ 39'\ 25''.04 & +(4)+(5) \\
\text{“} \quad SOL & 78^\circ\ 27'\ 06''.06 & +(7)+(8) \\
\text{“} \quad OLS & 30^\circ\ 53'\ 30''.81 & +(9) \\
\hline
 & 180^\circ\ 00'\ 01''.91 & \\
 & 180^\circ\ 00'\ 00''.37 & \\
\end{array}
$$

$$0 = 1''.54 + (4) + (5) + (7) + (8) + (9)$$

(c) Quadrilateral N. Base, S. Base, Oneota, Lester.

$$\frac{\sin LNS}{\sin LNO}\,\frac{\sin LSO}{\sin NSL}\,\frac{\sin LON}{\sin LOS} = 1$$

$$
\begin{array}{ll}
LNS = 113^\circ\ 39'\ 04''.43 + (2) & LNO = 124^\circ\ 09'\ 40''.05 + (1) \\
LSO = 70^\circ\ 39'\ 25''.04 + (4) + (5) & NSL = 47^\circ\ 31'\ 19''.91 + (5) \\
LON = 43^\circ\ 46'\ 26''.40 + (8) & LOS = 78^\circ\ 27'\ 06''.06 + (7) + (8)
\end{array}
$$

$$
\begin{array}{ll}
9.9618975,6 - 9,22\ (2) & 9.9177479,3 - 14,29\ (1) \\
9.9747660,1 + 7,39\,\{(4)+(5)\} & 9.8677849,8 + 19,28\ (5) \\
9.8399903,4 + 21,98\ (8) & 9.9911180,3 + 4,30\,\{(7)+(8)\} \\
\overline{539,1} & \overline{509,4} \\
509,4 & \\
\overline{29,7} &
\end{array}
$$

Check by deducting $\frac{1}{3}$ of the spherical excesses of the triangles from the angles.

113° 39′ 04″.36	124° 09′ 40″.01
70° 39′ 24″.92	47° 31′ 19″.84
43° 46′ 26″.36	78° 27′ 05″.93
9.9618976,2	9.9177479,9
9.9747659,3	9.8677848,6
9.8399902,5	9.9911179,8
38,0	8,3
8,3	
29,7	

The two methods agree well.

A glance at the log. differences for 1″ shows that by expressing them in units of the sixth place of decimals their average value is unity nearly. We have, then, for the side equation,

$$1.43\,(1) - 0.92\,(2) + 0.74\,(4) - 1.19\,(5) - 0.43\,(7) + 1.77\,(8) + 2.97 = 0$$

The Weight Equations.

$$(1) = -0.233\,\boxed{1} + 0.267\,\boxed{2}$$
$$(2) = +0.267\,\boxed{1} - 0.233\,\boxed{2}$$
$$(4) = +0.038\,\boxed{4} - 0.021\,\boxed{5}$$
$$(5) = -0.021\,\boxed{4} + 0.088\,\boxed{5}$$
$$(7) = +0.032\,\boxed{7}$$
$$(8) = +1.000\,\boxed{8}$$
$$(9) = +0.125\,\boxed{9}$$

The Correlate Equations.

	I.	II.	III.	Check.
$\boxed{1} =$	− 1		+ 1.43	− 0.43
$\boxed{2} =$	− 1		− 0.92	+ 1.92
$\boxed{4} =$	+ 1	+ 1	+ 0.74	− 2.74
$\boxed{5} =$		+ 1	− 1.19	+ 0.19
$\boxed{7} =$	+ 1	+ 1	− 0.43	− 1.57
$\boxed{8} =$		+ 1	+ 1.77	− 2.77
$\boxed{9} =$		+ 1		− 1.00

The check is formed by adding each horizontal row (Art. 84).

Expression of the Corrections in Terms of the Correlates.

	I.	II.	III.	Check.
	+ 0.233		− 0.333	+ 0.100
	− 0.267		− 0.246	+ 0.513
(1) =	− 0.034		− 0.579	+ 0.613
	− 0.267		+ 0.382	− 0.115
	+ 0.233		+ 0.214	− 0.447
(2) =	− 0.034		+ 0.596	− 0.562
	+ 0.038	+ 0.038	+ 0.028	− 0.104
		− 0.021	+ 0.024	− 0.004
(4) =	+ 0.038	+ 0.017	+ 0.052	− 0.108
	− 0.021	− 0.021	− 0.016	+ 0.058
		+ 0.088	− 0.105	+ 0.017
(5) =	− 0.021	+ 0.067	− 0.121	+ 0.075
(7) =	+ 0.032	+ 0.032	− 0.014	− 0.050
(8) =		+ 1.	+ 1.770	− 2.770
(9) =		+ 0.125		− 0.125

The Corrections in Terms of the Correlates (*collected*).

	I.	II.	III.
(1) =	− 0.034		− 0.579
(2) =	− 0.034		+ 0.596
(4) =	+ 0.038	+ 0.017	+ 0.052
(5) =	− 0.021	+ 0.067	− 0.121
(7) =	+ 0.032	+ 0.032	− 0.014
(8) =		+ 1.000	+ 1.770
(9) =		+ 0.125	

Formation of the Normal Equations.

	I.	II.	III.	Check.
− (1) =	+ 0.034		+ 0.579	− 0.613
− (2) =	+ 0.034		− 0.596	+ 0.562
+ (4) =	+ 0.038	+ 0.017	+ 0.052	− 0.108
+ (7) =	+ 0.032	+ 0.032	− 0.014	− 0.050
	+ 0.138	+ 0.049	+ 0.021	+ 0.209
(4) =		+ 0.017	+ 0.052	− 0.108
(5) =		+ 0.067	− 0.121	+ 0.075
(7) =		+ 0.032	− 0.014	− 0.050
(8) =		+ 1.	+ 1.770	− 2.770
(9) =		+ 0.125		− 0.125
	+ 0.049	+ 1.241	+ 1.687	− 2.978

	I.	II.	III.	Check.
+ 1.43(1)			+ 0.852	− 0.804
− 0.92(2)			+ 0.533	− 0.564
+ 0.74(4)			+ 0.038	− 0.080
− 1.19(5)			+ 0.144	− 0.089
− 0.43(7)			+ 0.006	+ 0.022
+ 1.77(8)			+ 3.133	− 4.903
	+ 0.021	+ 1.687	+ 4.706	− 6.418

The Normal Equations (*collected*).

I.	II.	III.
+ 0.138	+ 0.049	+ 0.021 = − 0.260
	+ 1.241	+ 1.687 = − 1.540
		+ 4.706 = − 2.970

The solution of these equations gives (page 295)

$$\text{I.} = -1.597$$
$$\text{II.} = -0.642$$
$$\text{III.} = -0.394$$

Substitute for I., II., III. their values in (4), and we have the general corrections.

Adding the local corrections and general corrections together, the total corrections to the measured angles result and are as follows:

	Local.	General.	Total.	p	pvv	Final Angles.
$x_1 =$	$-0''.64$	$-0''.18$	$=-0''.82$	2	1.34	124° 09′ 39″.87
$x_2 =$	$-0''.64$	$+0''.28$	$=-0''.36$	2	.26	113° 39′ 4″.71
$x_3 =$	$-0''.09$	$-0''.10$	$=-0''.19$	14	.50	122° 11′ 15″.42
$x_4 =$	$-0''.13$	$-0''.09$	$=-0''.22$	23	1.10	23° 08′ 5″.04
$x_5 =$	$-0''.50$	$+0''.04$	$=-0''.46$	6	1.27	47° 31′ 19″.95
$x_6 =$	$+0''.44$	$-0''.05$	$=+0''.39$	7	1.06	70° 39′ 24″.99
$x_7 =$		$-0''.07$	$=-0''.07$	31	.15	34° 40′ 39″.59
$x_8 =$		$-1''.33$	$=-1''.33$	1	1.77	43° 46′ 25″.07
$x_9 =$		$-0''.08$	$=-0''.08$	8	.05	30° 53′ 30″.73
					$[pvv] = 7.50$	

Number of local conditions $= 2$
Number of general conditions $= 3$

Total $= 5$

The method of solution just given is substantially the same as that employed on the survey of the Great Lakes between Canada and the United States by the U. S. Engineers.

The Precision of the Adjusted Values.

(a) To find the m. s. e. of an observation of weight unity.

Computation of $[pvv]$.

(1) From the preceding table $[pvv]$ has been found directly; thus

$$[pvv] = 7.50$$

(2) Check (Art. 114). From the station adjustments find $[v''v'']$

$$\begin{aligned} &\text{N. Base gives (p. 288) } 1.75 \\ &\text{S. Base gives (p. 289) } \underline{3.24} \\ &\qquad\qquad 4.99 = [v''v'']. \end{aligned}$$

From the general adjustment find $[ww]$.

$$\begin{aligned}
(\alpha)\ l_0' &\times \text{I.} = -0.26 \times -1.597 = +0.42 \\
l_0'' &\times \text{II.} = -1.54 \times -0.642 = +0.99 \\
l_0''' &\times \text{III.} = -2.97 \times -0.394 = +\underline{1.16} \\
& \qquad\qquad\qquad\qquad\qquad +2.57
\end{aligned}$$

$$\begin{aligned}
(\beta)\ l_0' &\times \frac{l_0'}{[aA]} = -0.26 \times -1.885 = +0.49 \\
l_0''.1 &\times \frac{l_0''1}{[bB.1]} = -1.45 \times -1.183 = +1.72 \\
l_0'''.2 &\times \frac{l_0'''2}{[cC.2]} = -0.94 \times -0.394 = +\underline{0.37} \\
& \qquad\qquad\qquad\qquad\qquad +2.58 \\
& \therefore [ww] \qquad\qquad = 2.58
\end{aligned}$$

$$\begin{aligned} \text{and } [pvv] &= 4.99 + 2.58 \\ &= 7.57 \end{aligned}$$

Hence taking the mean of the values of $[pvv]$,

$$\mu = \sqrt{\frac{7.54}{2+3}} = 1''.23$$

there being 2 local conditions and 3 net conditions.

(b) To find the m. s. e. of an angle in the adjusted figure.

$$\begin{aligned} \text{Angle} &= NLS \\ \therefore dF &= -(2) - (5) \\ &= +0.055\ \text{I.} - 0.067\ \text{II.} + 0.700\ \text{III.} \end{aligned}$$

from the weight equations.

From equations 25, Art. 114,

$$\begin{aligned} q_1 &= [\alpha\alpha]g_1 + [\alpha\beta]g_2 + \ldots \\ q_2 &= [\alpha\beta]g_1 + [\beta\beta]g_2 + \ldots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

The values of $[\alpha\alpha]$, $[\alpha\beta]$ . . . are given in the weight equations. Hence

$$q_1 = +0.267 \times 0 - 0.233 \times -1 = +0.233$$
$$q_2 = -0.233 \times 0 + 0.267 \times -1 = -0.267$$
$$q_3 = +0.038 \times 0 - 0.021 \times -1 = +0.021$$
$$q_4 = -0.021 \times 0 + 0.088 \times -1 = -0.088$$

g	q	gq
0	+ 0.233	0.
− 1	− 0.267	0.267
0	+ 0.021	0.
− 1	− 0.088	0.088

$$\frac{[gA]^2}{[aA]} = 0.022. \text{ (See the solution of the normal equations.)}$$
$$\frac{[gB.1]^2}{[bB.1]} = 0.006$$
$$\frac{[gC.2]^2}{[cC.2]} = 0.274$$
$$0.302$$

$$[gq] = 0.355$$
$$-0.302$$
$$u_F = +0.053$$
$$\therefore\ \mu_F = 1''.23 \sqrt{0.053}$$
$$= 0''.28$$

(c) To find the m. s. e. of a side in the adjusted figure.

Side = Oneota-Lester.

$$F = OL = NS \frac{\sin ONS \ \sin OSL}{\sin SON \ \sin OLS}$$

Therefore

$$dF = 1.33(1) + 1.33(2) + 0.74(4) + 0.74(5) - 3.04(7) - 3.52(9)$$

in units of the sixth place of decimals,

$$= -0.174 \text{ I.} - 0.475 \text{ II.} + 0.015 \text{ III.}$$

from the weight equations.

The solution is carried through exactly as in the preceding case. We find

$$[gq] = 2.011 \quad \text{and } u_F = 1.49$$

Hence
$$\mu \sqrt{u_F} = 1.23 \sqrt{1.49}$$
$$= 1.5 \text{ in units of the sixth place of decimals.}$$

Now
$$\log OL = 4.2189699 \qquad OL = 16556^{m}$$
$$\therefore \text{ m. s. e. of side} = \frac{16556}{0.434} \times 0.0000015$$
$$= 0^{m}.06$$

Solution of the Normal Equations.

I.	II.	III.	l	f (angle)
+ 0.138	+ 0.049	+ 0.021	− 0.260	+ 0.055
	+ 1.241	+ 1.687	− 1.540	− 0.067
		+ 4.706	− 2.970	+ 0.700
+ 1.	+ 0.355	+ 0.152	− 1.885	+ 0.399
	+ 1.224	+ 1.680	− 1.448	− 0.087
		+ 4.703	− 2.930	+ 0.692
				+ 0.022
	+ 1.	+ 1.373	− 1.183	− 0.071
		+ 2.396	− 0.945	+ 0.811
				+ 0.006
		+ 1.	− 0.394	+ 0.338
				+ 0.274

Ex. 1. Adjust the observed differences of longitude* given in the following table:

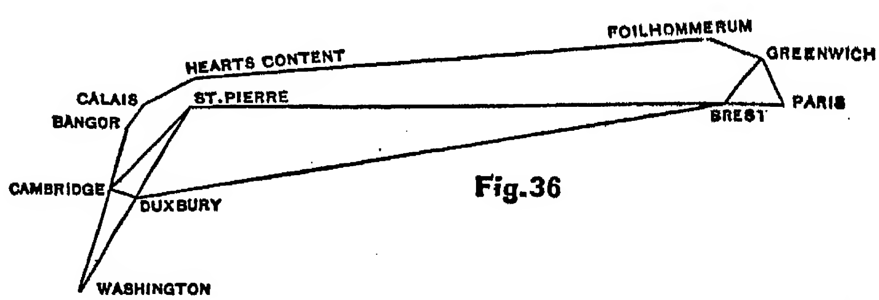


Fig.36

Dates.		Observed Differences. *h.*	*m.*	*s.*	*s.*	Corrections.
1851	Cambridge–Bangor,	0	9	23.080	± 0.043	v_1
1857	Bangor–Calais,		6	00.316	± 0.015	v_2
1866	Calais–Heart's Content,		55	37.973	± 0.066	v_3
1866	Heart's Content–Foilhommerum,	2	51	56.356	± 0.029	v_4
1866	Foilhommerum–Greenwich,		41	33.336	± 0.049	v_5
1872	Brest–Greenwich,		17	57.598	± 0.022	v_6
1872	Brest–Paris,		27	18.512	± 0.027	v_7
1872	Greenwich–Paris,		9	21.000	± 0.038	v_8
1872	St. Pierre–Brest,	3	26	44.810	± 0.027	v_9
1872	Cambridge–St. Pierre,		59	48.608	± 0.021	v_{10}
1869–1870	Cambridge–Duxbury,		1	50.191	± 0.022	v_{11}
1870	Duxbury–Brest,	4	24	43.276	± 0.047	v_{12}
{ 1867 1872	Washington–Cambridge,		23	41.041	± 0.018	v_{13}
1872	Washington–St. Pierre,	1	23	29.553	± 0.027	v_{14}

* *Coast and Geodetic Survey Report*, 1880, app. No. 6.

[Number of conditions $= n - s + 1$, where n is number of observed differences of longitude, and s is number of longitude stations.

The condition equations are

$$\begin{aligned} -v_6 + v_7 - v_8 &= +\overset{s.}{0.086} \\ -v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_9 + v_{10} &= +0.045 \\ -v_9 - v_{10} + v_{11} + v_{12} &= -0.049 \\ +v_{10} + v_{13} - v_{14} &= -0.096 \end{aligned}$$

The weights are taken inversely as the squares of the p. e.

Solution by method of correlates, as in Art. 110.]

Ex. 2. The system of triangulation shown in the figure was executed by Koppe in the determination of the axis (Airolo - Göschenen) of the St. Gothard tunnel.* In the following table the adjusted values are given side by side with the measured values. It is proposed as a problem of adjustment.

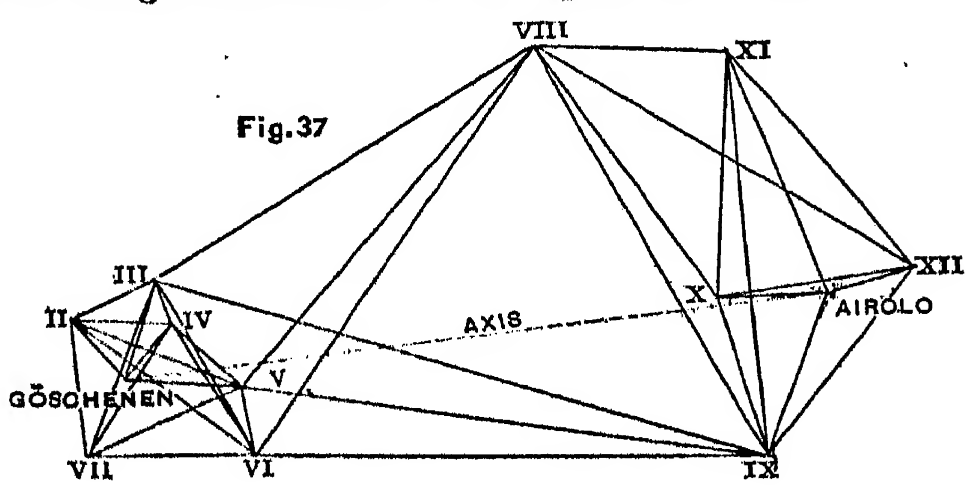


Fig.37

At Göschenen.

	Measured.			Adjusted.
II.	0°	00′	00″.00	00″.00
III.	44°	33′	10″.88	10″.03
IV.	69°	30′	12″.51	11″.62
V.	124°	58′	4″.23	5″.13

At II.

III.	0°	00′	00″.00	00″.00
IV.	37°	53′	54″.33	52″.97
V.	60°	29′	33″.13	33″.82
VI.	77°	4′	5″.67	8″.17
Göschenen	93°	11′	41″.69	40″.57
VII.	124°	16′	33″.98	33″.27

At III.

VIII.	0°	00′	00″.00	00″.00
IX.	53°	58′	14″.48	15″.49
VI.	99°	47′	50″.21	50″.86
IV.	102°	32′	51″.36	51″.90
Göschenen	138°	44′	28″.81	29″.70
VII.	144°	28′	12″.47	11″.40
II.	180°	59′	38″.94	39″.11

At IV.

	Measured.			Adjusted.
V.	0°	00′	00″.00	00″.00
VI.	15°	41′	3″.57	6″.29
VII.	74°	12′	20″.55	19″.86
Göschenen	80°	32′	48″.99	50″.12
II.	135°	44′	49″.77	50″.91
III.	199°	24′	11″.56	10″.73

At V.

IV.	0°	00′	00″.00	00″.00
VIII.	78°	40′	5″.91	6″.72
IX.	140°	44′	43″.51	44″.45
VI.	215°	32′	45″.41	43″.45
VII.	286°	19′	25″.30	27″.21
Göschenen	316°	00′	44″.92	43″.61
II.	338°	20′	33″.53	31″.74

At VII.

II.	0°	00′	00″.00	00″.00
III.	19°	11′	58″.44	59″.03
IV.	32°	4′	49″.32	48″.68
V.	64°	11′	54″.08	56″.05
VI.	90°	05′	39″.47	37″.00

* *Zeitschr. für Vermess.*, vol. iv.

At VIII.

XI.	0°	00′	00″.00	00″.00
XII.	18°	56′	17″.43	17″.54
X.	43°	50′	24″.03	24″.70
IX.	50°	18′	22″.52	20″.27
VI.	106°	30′	15″.04	15″.37
V.	112°		28″.72	29″.24
III.	130°	11′	30″.81	41″.54

At IX.

VI.	0°	00′	00″.00	00″.00
V.	8°	28′	17″.13	15″.06
III.	18°	33′	3″.27	5″.00
VIII.	63°	41′	28″.63	28″.55
X.	76°	59′	50″.89	51°.48
XI.	79°	10′	36″.33	36″.34
Airolo	109°	45′	39″.23	39″.33
XII.	123°	16′	23″.76	24″.23

At X.

XII.	0°	00′	00″.00	00″.00
Airolo	9°	49′	30″.02	37″.92
IX.	91°	30′	5″.16	5″.96
VIII.	252°	43′	46″.75	47″.49
XI.	275°	12′	8″.44	9″.74

At XI.

XII.	0°	00′	00″.00	00″.00
Airolo	16°	55′	55″.06	54″.38
IX.	37°	13′	59″.79	58″.43
VIII.	152°	26′	30″.24	30″.44

At XII.

IX.	0°	00′	00″.00	00″.00
Airolo	30°	31′	2″.30	3″.39
X.	42°	13′	20″.53	21″.33
VIII.	90°	3′	2″.22	1″.74
XI.	98°	40′	14″.95	13″.72

At Airolo.

XI.	0°	00′	00″.00	00″.00
XII.	94°	54′	56″.06	55″.26
IX.	230°	53′	7″.51	6″.98
X.	296°	26′	49″.43	51″.11

The distance X–XII. is $4416^m.8$.
There are 19 angle equations and 15 side equations in the adjustment.

SOLUTION BY SUCCESSIVE APPROXIMATION.

141. The rigorous forms of solution which have been given are suitable for a primary triangulation where the greatest precision is required. In secondary or tertiary work it would not be advisable to spend so much labor in the reduction, since the systematic errors remaining would probably largely outweigh the accidental errors eliminated. For work of this kind the method of solution by successive approximation is to be preferred.

The principle underlying the process of solution is that explained in Art. 115. Each condition or set of conditions

is adjusted for independently in succession, the values of the corrections found at each adjustment being closer and closer approximations to the final values. Should the values found after going through all of the conditions not satisfy the first, second, . . . groups of condition equations closely enough, the process must be repeated till the required accuracy is attained.

To make the operation as simple as possible let us take but a single condition at a time.

(1) Local equation at N. Base,

$$v_1 + v_2 + v_3 + 1.37 = 0$$

The solution is given in Ex. 1, Art. 121,

$$v_1 = -0''.64, \quad v_2 = -0''.64, \quad v_3 = -0''.09$$

(2) Local equation at S. Base,

$$v_4 + v_5 - v_8 + 1.07 = 0$$

The solution is given in Ex. 2, Art. 121,

$$v_4 = -0''.13, \quad v_5 = -0''.50, \quad v_8 = +0''.44$$

(3) Angle equation,

$$v_3 + v_4 + v_7 + 0.48 = 0$$

Using the values of v_3, v_4 already found as first approximations, the equation reduces to

$$v_3 + v_4 + v_7 + 0.26 = 0$$

The method of solution is given in Ex. 2, Art. 110,

$$v_3 = -0''.13, \quad v_4 = -0''.08, \quad v_7 = -0''.05$$

The successive approximations found so far, when added, give

$$v_1 = -0''.64 \qquad v_5 = -0''.55$$
$$v_2 = -0''.64 \qquad v_6 = +0''.44$$
$$v_3 = -0''.22 \qquad v_7 = -0''.05$$
$$v_4 = -0''.21$$

Proceed similarly with the remaining two condition equations. The resulting values will agree closely with the rigorous values already found.

142. In order to bring out still more clearly the advantages of solving in this way, let us take a more extended example. A good one is furnished by the triangulation (1874–1878) of the east end of Lake Ontario, omitting the system around the Sandy Creek base.

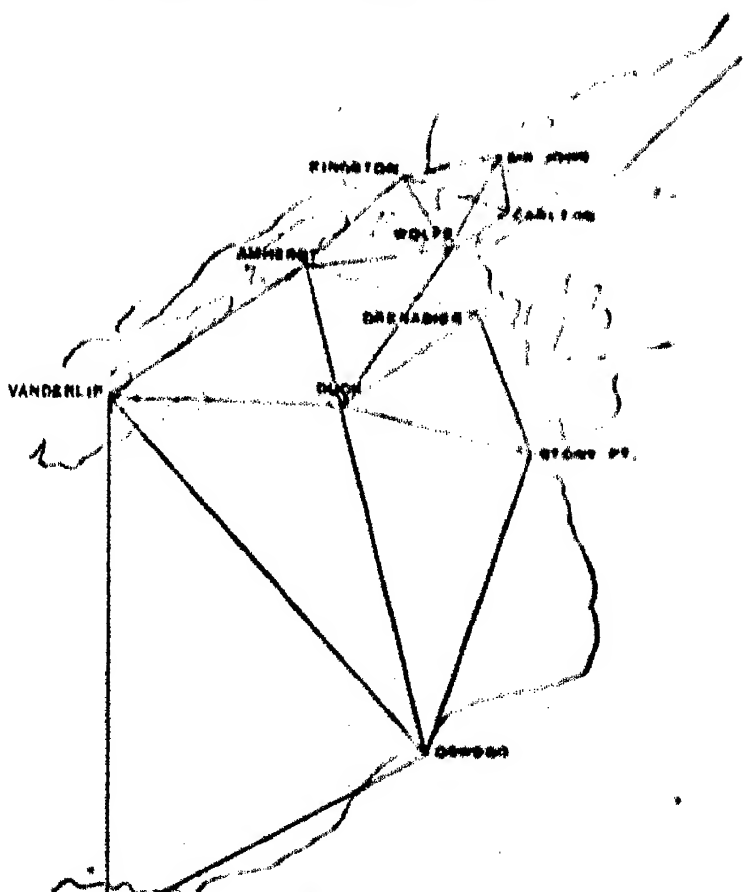


Fig. 38

The measured values of the angles are given in the following table. Each angle is taken to be of the same weight. In the last column are given the locally corrected angles found by the rigorous methods of solution.

Station occupied.	Angle as measured between		Locally corr. angles.
Sir John,	**Carlton and Kingston,**	90° 17′ 44″.91	
	Wolfe and Kingston,	56° 24′ 09″.77	
Carlton,	**Wolfe and Sir John,**	120° 48′ 06″.54	
	Kingston and Sir John,	62° 03′ 27″.56	
Kingston,	**Sir John and Wolfe,**	64° 40′ 59″.91	
	Carlton and Wolfe,	37° 02′ 04″.43	
	Wolfe and Amherst,	88° 19′ 14″.70	
Wolfe,	**Duck and Carlton,**	188° 07′ 18″.54	
	Amherst and Carlton,	140° 12′ 34″.44	
	Kingston and Carlton,	84° 13′ 14″.34	
	Sir John and Carlton,	25° 18′ 16″.80	
Amherst,	**Kingston and Wolfe,**	35° 41′ 23″.02	22″.69
	Kingston and Duck,	111° 45′ 28″.46	28″.68
	Wolfe and Duck,	76° 04′ 06″.32	05″.99
	Grenadier and Duck,	54° 38′ 00″.34	
	Duck and Vanderlip,	71° 15′ 25″.43	25″.32
	Vanderlip and Kingston,	176° 59′ 06″.11	06″.00
Duck,	**Oswego and Vanderlip,**	104° 08′ 58″.93	59″.10
	Vanderlip and Amherst,	70° 26′ 31″.99	32″.16
	Amherst and Wolfe,	56° 01′ 12″.47	12″.64
	Wolfe and Grenadier,	18° 45′ 43″.36	43″.53
	Grenadier and Stony Pt.,	49° 53′ 12″.77	12″.94
	Stony Pt. and Oswego,	60° 44′ 19″.46	19″.63
Grenadier,	**Stony Pt. and Duck,**	78° 13′ 33″.64	33″.84
	Duck and Amherst,	50° 35′ 04″.28	04″.19
	Duck and Stony Pt.,	281° 46′ 25″.89	26″.16
	Amherst and Stony Pt.,	231° 11′ 22″.04	21″.97
Stony Pt.,	**Oswego and Duck,**	88° 22′ 00″.86	
	Duck and Grenadier,	51° 53′ 12″.60	12″.70
	Grenadier and Duck,	308° 06′ 47″.21	47″.30
Oswego,	**Sodus and Vanderlip,**	80° 29′ 46″.10	46″.59
	Sodus and Duck,	107° 19′ 03″.28	03″.96
	Sodus and Stony Pt.,	138° 12′ 49″.44	48″.28
	Vanderlip and Duck,	26° 49′ 16″.61	17″.37
	Vanderlip and Stony Pt.,	57° 43′ 01″.96	01″.69
	Duck and Stony Pt.,	30° 53′ 42″.88	44″.32
Vanderlip,	**Amherst and Duck,**	38° 18′ 07″.12	07″.30
	Amherst and Oswego,	87° 19′ 53″.47	53″.16
	Duck and Oswego,	49° 01′ 45″.54	45″.86
	Duck and Sodus,	87° 59′ 12″.55	12″.42
	Oswego and Sodus,	38° 57′ 26″.55	26″.56
	Sodus and Amherst,	233° 42′ 40″.41	40″.28
Sodus,	**Vanderlip and Oswego,**	60° 32′ 57″.55	

The local and general equations are formed as usual (see Arts. 117–140). The general rule in the solution is to adjust for one condition at a time. Instead, however, of following out this rule strictly, it is often better to adjust for a *group* of conditions simultaneously. Often a group is almost as easily managed as a single condition. No rule can be given to cover all cases, and much must be left to the judgment and ingenuity of the computer.

143. **The Local Adjustment at Each Station.**—

(a) Adjust for each sum angle separately.

Rule and example in Art. 121.

(b) Adjust for closure of the horizon.

Rule and example in Art. 121.

At stations Sir John, Carlton, Kingston, Wolfe there are no local conditions, and at each of the stations Amherst, Stony Point, Sodus there is one angle independent of the others, and therefore not locally adjusted.

The angles at station Amherst may be rigorously adjusted, as in Art. 121. The resulting values are given in the table. If we break the adjustment into two parts, as in (a) and (b), we have:

(a) *Sum Angle.*

	Measured values.	Adjusted.
Kingston–Wolfe,	35° 41′ 23″.02 − 0″.29	22″.73
Wolfe–Duck,	76° 04′ 06″.32 − 0″.29	06″.03
	111° 45′ 29″.34	28″.76
Kingston–Duck,	111° 45′ 28″.46 + 0″.29	28″.75 check.
	3)0″.88	
	0″.29	

(b) *Closure of Horizon.*

Kingston–Wolfe,	35° 41′ 22″.73 − 0″.08	22″.65
Wolfe–Duck,	76° 04′ 06″.03 − 0″.07	05″.96
Duck–Vanderlip,	71° 15′ 25″.43 − 0″.08	25″.35
Vanderlip–Kingston,	176° 59′ 06″.11 − 0″.07	06″.04
	4)00″.30	00″.00 check.
	00″.075	

The adjusted values agree closely with those from the simultaneous solution, as given in the table.

At station Duck the angles close the horizon. Hence the correction to each angle is one-sixth of the difference of their sum from 360°. (See Art. 121.)

144. **The General Adjustment.**—The local adjustment being finished, we shall consider the adjusted angles to be independent of one another and to be of the same weight. We are therefore at liberty to break up the net into its simplest parts. We have in our figure, first a quadrilateral *SCWK*, next two single triangles *KWA*, *AWD*, next a central polygon *DAGSOV*, and, lastly, a single triangle *VOS*. These three figures include most cases that arise in any triangulation net.

(a 1) *Adjustment of a Quadrilateral.*—In the quadrilateral *SCKW* all of the eight angles 1, 2, . . . 8 are supposed to be equally well measured.

(1) The Angle Equations.

The angle equations from the triangles *SCW*, *CWK*, *WKS* may be written in general terms

$$v_1 + v_2 + v_3 + v_4 = l_1$$
$$v_3 + v_4 + v_5 + v_6 = l_2$$
$$v_5 + v_6 + v_7 + v_8 = l_3$$

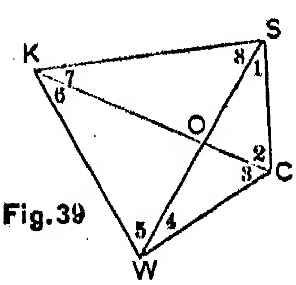


Fig. 39

As these equations are entangled, if we adjusted for each in succession a great many repetitions of the adjustment would be necessary to obtain values that would satisfy the equations simultaneously. It is, therefore, better to adjust simultaneously, and it happens that a very simple rule for doing this can be found.

Call k_1, k_2, k_3, k_4 the correlates of the equations in order; then the correlate equations are

$$\begin{aligned} k_1 \quad &= v_1 & \qquad k_2 + k_3 &= v_5 \\ k_1 \quad &= v_2 & k_2 + k_3 &= v_6 \\ k_1 + k_2 &= v_3 & k_3 &= v_7 \\ k_1 + k_2 &= v_4 & k_3 &= v_8 \end{aligned}$$

and the normal equations

$$\begin{aligned} 4k_1 + 2k_2 \qquad\quad &= l_1 \\ 2k_1 + 4k_2 + 2k_3 &= l_2 \\ 2k_2 + 4k_3 &= l_3 \end{aligned}$$

Solving these equations, there result

$$\begin{aligned} k_1 &= \tfrac{1}{8}(+3l_1 - 2l_2 + l_3) \\ k_2 &= \tfrac{1}{8}(-2l_1 + 4l_2 - 2l_3) \\ k_3 &= \tfrac{1}{8}(+l_1 - 2l_2 + 3l_3) \end{aligned}$$

Substitute these values in the correlate equations, and

$$\begin{aligned} v_1 = v_2 &= \tfrac{1}{8}(+3l_1 - 2l_2 + l_3) \\ v_3 = v_4 &= \tfrac{1}{8}(+l_1 + 2l_2 - l_3) \\ v_5 = v_6 &= \tfrac{1}{8}(-l_1 + 2l_2 + l_3) \\ v_7 = v_8 &= \tfrac{1}{8}(+l_1 - 2l_2 + 3l_3) \end{aligned}$$

which may be written

$$\begin{aligned} v_1 = v_2 &= \tfrac{1}{4}l_1 - \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1) \\ v_3 = v_4 &= \tfrac{1}{4}l_1 + \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1) \\ v_5 = v_6 &= \tfrac{1}{4}l_3 + \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1) \\ v_7 = v_8 &= \tfrac{1}{4}l_3 - \tfrac{1}{4}(l_2 - \tfrac{1}{2}l_3 - \tfrac{1}{2}l_1) \end{aligned}$$

whence follows at once the convenient rule for adjusting the quadrilateral, so far as the angle equations are concerned:

(α) *Write the measured angles in order of azimuth in two sets of four each, the first set being the angles of SCW, and the second those of WKS.*

(β) *Adjust the angles of each set by one-fourth of the difference of this sum from* 180° + *excess of triangle, arranging the adjusted angles in two columns, so that the first column will show the angles of SCK, and the second those of CWK.*

(γ) *Adjust the first column by one-fourth of the difference of its sum from* 180° + *excess of triangle, and apply the same correction, with the sign changed, to the second column.*

The spherical excesses of the triangles SCW, CWK, WKS being 0″.16, 0″.35, and 0″.47 respectively, the adjustment of the quadrilateral may be arranged as follows:

	Measured angles.			Adjusted angles.
	33° 53′ 35″.14	35″.56		35″.40
	62° 03′ 27″.56	27″.98		27″.82
	58° 44′ 38″.98		39″.40	39″.56
	25° 18′ 16″.80		17″.22	17″.38
	179° 59′ 58″.48			00″.16 check
$180 + \varepsilon =$	180° 00′ 00″.16			
	4)1″.68			
	0″.42			
	58° 54′ 57″.54		58″.11	58″.27
	37° 02′ 04″.43		04″.99	05″.15
	27° 38′ 46″.48	47″.04		46″.88
	56° 24′ 9″.77	10″.33		10″.17
	179° 59′ 58″.22	00″.91		00″.47 check
$180 + \varepsilon =$	180° 00′ 00″.47	00″.28		
	4)2″.25	4)0″.63		
	0″.56	0″.16		

(2) The Side Equation.

Using the values of the angles just found, we next form the side equation with pole at O. It is

$$\frac{\sin OSC}{\sin SCO}\,\frac{\sin OCW}{\sin CWO}\,\frac{\sin OWK}{\sin WKO}\,\frac{\sin OKS}{\sin KSO} = 1$$

or writing it in general terms when reduced to the linear form (see Art. 129)

$$a_1v_1' + a_2v_2' + a_3v_3' + a_4v_4' + a_5v_5' + a_6v_6' + a_7v_7' + a_8v_8' = l_4$$

where v_1', v_2', . . . are the corrections resulting from the side equation.

Solving as in Ex. 2, Art. 110, we have the corrections

$$v_1' = \frac{a_1}{[aa]}l_4,\quad v_2' = \frac{a_2}{[aa]}l_4,\quad \ldots$$

These corrections may be found still more rapidly as follows: Since the side equation may be so transformed that the coefficients $a_1, a_2, \ldots$ are approximately equal to unity numerically (see Art. 131), we may take each of them to be unity, and then

$$v_1' = v_3' = v_5' = v_7' = +\tfrac{1}{8}\, l_4$$
$$v_2' = v_4' = v_6' = v_8' = -\tfrac{1}{8}\, l_4$$

that is, *the corrections to the angles are numerically equal, but are alternately + and —.*

This plan has the additional advantage of not disturbing the angle equations.

Returning to our numerical example, we first reduce the side equation to the linear form.

$$\begin{aligned} OSC &= 33^\circ\ 53'\ 35''.40 + v_1 \\ OCW &= 58^\circ\ 44'\ 39''.56 + v_3 \\ OWK &= 58^\circ\ 54'\ 58''.27 + v_5 \\ OKS &= 27^\circ\ 38'\ 46''.88 + v_7 \end{aligned} \qquad \begin{aligned} SCO &= 62^\circ\ 03'\ 27''.82 + v_2 \\ CWO &= 25^\circ\ 18'\ 17''.38 + v_4 \\ WKO &= 37^\circ\ 02'\ 05''.15 + v_6 \\ KSO &= 56^\circ\ 24'\ 10''.17 + v_8 \end{aligned}$$

$$\begin{array}{l} 9.7463587 + 31.3\, v_1 \\ 9.9318952 + 12.8\, v_3 \\ 9.9326832 + 12.7\, v_5 \\ 9.6665301 + 40.2\, v_7 \\ \hline \quad 72 \\ \quad 70 \\ \hline \quad 2 \end{array} \qquad \begin{array}{l} 9.9461673 + 11.2\, v_2 \\ 9.6308691 + 44.5\, v_4 \\ 9.7797125 + 27.9\, v_6 \\ 9.9206181 + 14.0\, v_8 \\ \hline \quad 70 \end{array}$$

Dividing by 20, which will reduce the coefficients to unity approximately, and

$$1.56 v_1' - 0.56 v_2' + 0.64 v_3' - 2.22 v_4' + 0.64 v_5' - 1.40 v_6' + 2.01 v_7' - 0.70 v_8' + 0.10 = 0$$

Hence

$$[aa] = 15$$

and

$$v_1' = -0''.01,\ v_2' = 0''.00,\ v_3' = 0''.00,\ v_4' = +0''.01, \text{ etc.}$$

By the second rule the corrections would be $\mp \dfrac{0''.1}{8}$, that is, $\mp 0''.01$ alternately, which values differ but little from the preceding.

The total corrections to the angles are the sums of the two sets of corrections from the angle and side equations.

(a 2) *Adjustment of a Quadrilateral.*—By the following artifice the quadrilateral may be *rigorously* adjusted for the side equation without disturbing the angle equation adjustment, which amounts to the same thing as the simultaneous adjustment of the angle and side equations.

Suppose that the angle equations have been adjusted as already explained in (a 1). If $v_1', v_2', \ldots v_8'$ denote the corrections arising from the side equation, the condition equations may be written

$$\begin{gathered}
v_1' + v_2' + v_3' + v_4' = 0 \\
v_3' + v_4' + v_5' + v_6' = 0 \\
v_5' + v_6' + v_7' + v_8' = 0 \\
a_1 v_1' + a_2 v_2' + a_3 v_3' + a_4 v_4' + a_5 v_5' + a_6 v_6' + a_7 v_7' + a_8 v_8' = l_4'
\end{gathered}$$

By writing the corrections in the form

$$\begin{aligned}
v_1' &= +v + v' & \qquad v_5' &= +v + v''' \\
v_2' &= +v - v' & v_6' &= +v - v''' \\
v_3' &= -v + v'' & v_7' &= -v + v'''' \\
v_4' &= -v - v'' & v_8' &= -v - v''''
\end{aligned}$$

the first three condition equations become $0 = 0$ identically, and we have therefore to deal only with the single condition equation

$$\begin{aligned}
(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)v + (a_1 - a_2)v' \\
+ (a_3 - a_4)v'' + (a_5 - a_6)v''' + (a_7 - a_8)v'''' = l_4'
\end{aligned}$$

with

$$(v + v')^2 + (v - v')^2 + (-v + v'')^2 + (-v - v'')^2 + \ldots = \text{a min.}$$

The correlate equations are

$$\begin{aligned}
(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)k &= 4v \\
(a_1 - a_2)k &= v' \\
(a_3 - a_4)k &= v'' \\
(a_5 - a_6)k &= v''' \\
(a_7 - a_8)k &= v''''
\end{aligned}$$

Substitute in the condition equation, and

$$k\left\{\tfrac{1}{4}(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)^2 + (a_1 - a_2)^2 \right.$$
$$\left. + (a_3 - a_4)^2 + (a_5 - a_6)^2 + (a_7 - a_8)^2\right\} = l_4'$$

from which k can be found.

Hence the corrections are known.

The complete adjustment of our quadrilateral is contained in the following table:

Meas. Angles.		Local Angles.	Log. Sines.		Log. Diff.		Sums.	Sqs.
33° 53′ 35″.14	35″.56	35″.40	9.7463587		+ 31.3			
62° 03′ 27″.56	27″.98	27″.82		9.9461673		+ 11.2	42.5	1806
58° 44′ 38″.98	39″.40	39″.56	9.9318952			+ 12.8		
25° 18′ 16″.80	17″.22	17″.38		9.6308691	+ 44.5		57.3	3283
58″.48								
00″.16								
1″.68								
58° 54′ 57″.54	58″.11	58″.27	9.9326832		+ 12.7			
37° 02′ 04″.43	04″.99	05″.15		9.7798125		+ 27.9	40.6	1648
27° 38′ 46″.48	47″.04	46″.88	9.6665301			+ 40.2		
56° 24′ 09″.77	10″.33	10″.17		9.9206181	+ 14.0		54.2	2938
58″.22	00″.91		72	70	102.5	92.1		
00″.47	00″.28		70		92.1			
2″.25	0″.63		2		4)10.4			
					2.6		5.2	27
								9702

$$\therefore \frac{v}{2.6} = \frac{v'}{42.5} = \frac{v''}{57.3} = \frac{v'''}{40.6} = \frac{v''''}{54.2} = -\frac{2}{9702}$$

Hence the corrections are known. These corrections, applied to the local angles, give the final angles required.

(b) *Adjustment of a Single Triangle.*

Rule and example in Ex. 2, Art. 109, and Ex. 2, Art. 110.

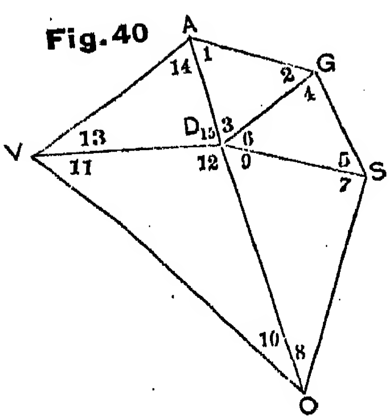
The single triangles in our figure are Kingston, Wolfe, Amherst; Wolfe, Duck, Amherst; Vanderlip, Oswego, Sodus.

For example, take the first ($\varepsilon = 0''.72$)

	Measured.			Adjusted.
Kingston,	88°	19′	14″.70	15″.78
Wolfe,	55°	59′	20″.10	21″.18
Amherst,	35°	41′	22″.69	23″.77
	179°	59′	57″.49	0″.73 check.
$180 + \varepsilon =$	180°	00′	00″.72	
			3)3″.23	
			1″.08	

(c) *Adjustment of a Central Polygon.*—In the central polygon Duck, Amherst, Grenadier, Stony Point, Oswego, Vanderlip the condition equations in general terms are:

Fig. 40



Local equation (horizon equation),

$$v'_3 + v'_6 + v'_9 + v'_{12} + v'_{15} = l_1$$

Angle equations,

$$v'_1 + v'_2 + v'_3 = l_2$$
$$v'_4 + v'_5 + v'_6 = l_3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$v'_{13} + v'_{14} + v'_{15} = l_6$$

Side equation (pole at Duck),

$$a_1 v'_1 + a_2 v'_2 + a_4 v'_4 + a_5 v'_5 + \ . \ . \ . + a_{14} v'_{14} = l$$

We may adjust for these equations in order, first the horizon equation, then the angle equations separately, as they are not entangled, and next the side equation.

A rigorous adjustment may, however, be carried out at once with very little additional labor. Adjust first each angle equation by itself, and let (v'_1), (v'_2), . . . be the values that result. Let (1), (2), . . . denote the farther corrections to the measured angles in order arising from the local and side equations, so that

$$v'_1 = (v'_1) + (1)$$
$$v'_2 = (v'_2) + (2)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

If we substitute these values in the above equations we have the new condition equations

$$
\begin{aligned}
a_1(1) + a_2(2) + \dots + a_{14}(14) &= l \\
(3) + (6) + (9) + (12) + (15) &= l' \\
(1) + (2) + (3) &= 0 \\
(4) + (5) + (6) &= 0 \\
\cdot \qquad \cdot \qquad \cdot & \\
(13) + (14) + (15) &= 0
\end{aligned}
$$

from which to find (1), (2), (3), . . . (15).

Calling k_1, k_2, I, II, . . . the correlates of the condition equations in order, we have the correlate equations

$$
\begin{aligned}
(1) &= a_1k_1 + I & (4) &= a_4k_1 + II & \dots \\
(2) &= a_2k_1 + I & (5) &= a_5k_1 + II & \dots \\
(3) &= k_2 + I & (6) &= k_2 + II & \dots
\end{aligned}
$$

Eliminate now the angle equation correlates. By addition

$$
\begin{aligned}
(a_1 + a_2)k_1 + 3I + k_2 &= 0 \\
(a_4 + a_5)k_1 + 3II + k_2 &= 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &
\end{aligned}
$$

Hence

$$
\begin{aligned}
(1) &= +\tfrac{1}{3}(2a_1 - a_2)k_1 - \tfrac{1}{3}k_2 \\
(2) &= -\tfrac{1}{3}(a_1 - 2a_2)k_1 - \tfrac{1}{3}k_2 \\
(3) &= -\tfrac{1}{3}(a_1 + a_2)k_1 + \tfrac{2}{3}k_2 \\
\cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

Substitute these values of (1), (2), . . . in the condition equations, and we have the normal equations

$$
\begin{aligned}
2\{[aa] - a_1a_2 - a_4a_5 - \dots\}k_1 - [a]k_2 &= 3l \\
-[a]k_1 + 10k_2 &= 3l'
\end{aligned}
$$

Solving, we find k_1, k_2, and thence (1), (2), . . . are known.

The normal equations may be written down directly in every case, as the law of their formation is as evident as that of ordinary normal equations. They involve only two unknowns, k_1, k_2, no matter how many sides the polygon has.

It is evident that the elimination of the angle equation correlates $I, II, \ldots$ from the correlate equations before forming the normal equations amounts to the same thing as first forming the normal equations in the usual way and then eliminating $I, II, \ldots$ For the normal equations from the correlate equations are

$$
\begin{array}{lllllll}
[aa]k_1 & & + (a_1 + a_2)I & + (a_4 + a_5)\ II & + \ldots & = l' \\
 & + 5k_2 & + \ I & + \ II & + \ldots & = l'' \\
(a_1 + a_2)k_1 & + \ k_2 & + 3I & & & = 0 \\
(a_4 + a_5)k_1 & + \ k_2 & & + 3II & & = 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Substitute for $I, II, \ldots$ their values from the third, fourth, ... equations in the first two, and we find the two normal equations as before.

The introduction of the method of eliminating the correlates arising from one set of condition equations before forming the normal equations is due to Schleiermacher, of the Hessian survey. For a fuller account of it see Fischer's *Geodäsie*, part iii. p. 93; Hügel in *General Bericht der europäischen Gradmessung*, 1867, pp. 106 seq.; Nell in *Zeitschr. für Vermess.*, vol. x. pp. 1 seq.; vol. xii. pp. 313 seq.

The process is not of any special advantage except in such problems as that under discussion, and then it is better to use the final formula for the normal equations directly.

We shall now proceed with our numerical example.

At station Duck the measured values of the angles are taken, at the other stations the locally adjusted angles.

Given Angles.		Log. Sines.		Diff. 1″.	Squares.	Products.	Sums.
54° 38′ 00″.34	00″.62	9.9114060		14.9	222.0	257.8	2.4
50° 35′ 04″.10	04″.47		9.8879338	17.3	299.3		
74° 46′ 55″.83	56″.12						
180° 00′ 00″.36							
180° 00′ 01″.21 = 180 + ε							
3)0″.85							
0″.28							
78° 13′ 33″.84	34″.40	9.9907654		4.4	19.4	72.6	12.1
51° 53′ 12″.70	13″.35		9.8958618	16.5	272.2		
49° 53′ 12″.77	13″.42						
59″.31							
1″.26							
1″.95							
0″.65							
88° 22′ 00″.86	00″.47	9.9998235		0.6	.4	21.1	34.6
30° 53′ 44″.32	43″.93		9.7105188	35.2	1239.0		
60° 44′ 19″.46	19″.07						
04″.64							
3″.47							
1″.17							
0″.39							
26° 49′ 17″.37	18″.15	9.6543842		41.6	1730.6	761.3	+ 23.3
49° 01′ 45″.86	46″.64		9.8779750	18.3	334.9		
104° 08′ 58″.93	59″.70						
2″.16							
4″.49							
2″.33							
0″.78							
38° 18′ 07″.30	06″.33	9.7022537		26.7	712.9	189.6	+ 19.6
71° 15′ 25″.32	24″.36		9.9763353	7.1	50.4		
70° 26′ 31″.99	31″.02						
04″.61		6328	6247		4881.1	1302.4	− 6.2
1″.71		6247			1302.4		
2″.90		81			6183.5		
0″.97		3			2		
		243			12367.0		

The Normal Equations.

$$12367k_1 + 6.2k_2 = -243$$
$$6.2k_1 + 10k_2 = 2.01$$

$$\therefore\ k_1 = -0.020$$
$$k_2 = +0.213$$

Local Equation at Station Duck.

74°	46′	56″.12
49°	53′	13″.42
60°	44′	19″.07
104°	08′	59″.70
70°	26′	31″.02
359°	59′	59″.33
360°	00′	00″.00
		00″.67
		3
		2″.01

Corrections.	Adjusted Angles.
(1) = − 0″.39	54° 38′ 00″.23
(2) = + 0″.26	50° 35′ 04″.73
(3) = + 0″.13	74° 46′ 56″.25
(4) = − 0″.24	78° 13′ 34″.25
(5) = + 0″.18	51° 53′ 13″.53
(6) = + 0″.06	49° 53′ 13″.48
(7) = − 0″.31	88° 22′ 00″.16
(8) = + 0″.40	30° 53′ 44″.33
(9) = − 0″.09	60° 44′ 18″.98
(10) = − 0″.75	26° 49′ 17″.40
(11) = + 0″.45	49° 01′ 47″.09
(12) = + 0″.30	104° 08′ 60″.00
(13) = − 0″.47	38° 18′ 05″.86
(14) = + 0″.20	71° 15′ 24″.56
(15) = + 0″.27	70° 26′ 31″.29

Approximate Method of Finding the Precision.

145. An adjustment may be carried out rigorously so far as finding the values of the unknowns is concerned, but only an approximate value of the m. s. e. of the angles or sides may be thought necessary.

In good work the following method will give results nearly the same as those found by the rigorous process.

The average value μ' of the m. s. e. of an angle in a triangulation net after adjustment is easily seen from Art. 102 to be

$$\mu' = \sqrt{\frac{n - n_c}{n}}\,\mu$$

where

n = number of angles observed.
n_c = number of local and general conditions.
μ = m. s. e. of a measured angle of weight unity.

The value of μ is, by the usual formula,

$$\mu = \sqrt{\frac{[pvv]}{n_c}}$$

To find the m. s. e. of a side of a triangle a single chain of the best-shaped triangles between the base and the side is selected, all tie lines being rejected. Then, assuming the base to be exact and the m. s. e. of each adjusted angle to be μ', we have from Ex. 9, p. 234,

$$\mu^2_{\log a_n} = \tfrac{2}{3}\mu'^2 [\delta_A^2 + \delta_B^2 + \delta_A \delta_B]$$

where δ_A, δ_B are the log. differences corresponding to 1″ for the angles A, B in a table of log. sines.

A form still more approximate was used on the U. S. Lake Survey in the determination of the precision of a side of the primary triangulation. The angles of each triangle were taken to be independent of one another. In this case evidently (Ex. 5, p. 109)

$$\mu^2_{\log a_n} = \mu'^2 [\delta_A^2 + \delta_B^2]$$

The earlier work of the Coast Survey was computed from this same formula.

Ex. To find the m. s. e. of the side OL as derived from the base NS in the figure $ONSL$ (Fig. 19).

Number of angles measured = 9.

Number of conditions, local and general, = 5.

From the adjustment (Art. 140) $[pvv] = 7.54$.

$$\therefore \mu = \sqrt{\frac{7.54}{5}}$$

$$= 1''.23$$

$$\mu' = \sqrt{\frac{9-5}{9}}\,\mu$$

$$= \tfrac{2}{3}\mu$$

The chain of triangles is ONS, OLS.

Station.	Angles.	δ	Sqs.	Prods.
N. Base,	122° 11′ 15″	− 13.2	174.2	
S. Base,	23° 08′ 05″			− 401.3
Oneota,	34° 40′ 40″	+ 30.4	924.2	
S. Base,	70° 39′ 25″	+ 7.4	54.8	
Oneota,	78° 27′ 05″			260.5
Lester,	30° 53′ 30″	+ 35.2	1239.0	
			2251.4	

$$\mu_{\log OL} = \sqrt{2251.4 \times \tfrac{1}{6}}$$

$$= 31.6 \text{ in units of the seventh decimal place.}$$

Also $OL = 16556$ metres.

Hence μ_{OL} is known.

The Method of Directions.

146. This method is due to Bessel. Various modifications of Bessel's plan of making the observations are used on different surveys. The following is that used on the U. S. Coast Survey.

"In any set, after the objects have been observed in the order of graduation they are re-observed with instrument reversed in the opposite order; the mean of the two observations upon each object is then taken. The number of such sets and the number of positions made depends on the accuracy required and upon the perfection of the instrument." A single series of means is called an "arc."

"The direction instrument requires that it should be turned on its stand or changed in position, in order that the direction of any one line, and consequently of all, should fall upon different parts of the circle as the only security against errors of graduation. The number of positions

varies from five to twenty-one of nearly equal arcs; and in each position the circuit of the horizon is made, giving the direction of each line by two observations, one in the direct and the other in the reversed position of the telescope. These circuits or series are repeated in each position until two to five values of each direction are obtained. Each angle is therefore determined by from 35 to 63 measurements in the direct and a like number in the reversed position of the telescope."

147. **The Local Adjustment.**—Let O be the station occupied, and 1, 2, 3, . . . the stations sighted at in order of azimuth. Let some one direction, as $O1$, be selected as the zero direction, and let A, B, . . . denote the most probable values of the *angles* which the directions of the different signals make with this direction.

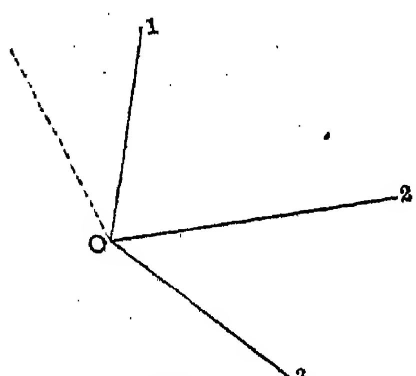


Fig. 41

In the first arc let X_1 denote the most probable value of the angle between the zero of the limb of the instrument and the direction of the signal taken as the zero direction; then if M_1', M_1'', . . . denote the readings of the limb for the different signals, and v_1', v_1'', . . . the most probable corrections to these readings, we have the observation equations

$$\begin{aligned} X_1 - M_1' &= v_1' \\ X_1 + A - M_1'' &= v_1'' \\ X_1 + B - M_1''' &= v_1''' \\ \ldots\quad \ldots\quad &\ldots \end{aligned}$$

The zero of the limb being changed in the next arc, we have in like manner

$$\begin{aligned} X_2 - M_2 &= v_2' \\ X_2 + A - M_2 &= v_2'' \\ X_2 + B - M_2 &= v_2''' \\ \ldots\quad \ldots\quad &\ldots \end{aligned}$$

and so on for the remaining arcs.

If now p_1', p_1'', . . .; p_2', p_2'', . . .; . . . denote the weights of the measured directions of the several series of arcs, the normal equations follow at once. They are

$$
\begin{array}{llll}
[p_1]X_1 & & +p_1''A+p_1'''B+\ldots & =[p_1M_1] \\
& [p_2]X_2 & +p_2''A+p_2'''B+\ldots & =[p_2M_2] \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot \\
p_1''X_1+p_2''X_2+\ldots & +[p'']A & & =[p''M''] \\
p_1'''X_1+p_2'''X_2+\ldots & & +[p''']B & =[p'''M'''] \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot
\end{array}
$$

from which the unknowns may be found.

In order to shorten the numerical work a course similar to that of Art. 41 may be followed. Let

$$
\begin{array}{ll}
X_1 = M_1' + x_1 & A = A' + (A) \\
X_2 = M_2' + x_2 & B = B' + (B) \\
\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}
$$

where M_1', M_2', . . . A', B', . . . are approximate values of X_1, X_2 . . . A, B, . . ., and x_1, x_2, . . . (A), (B), . . . denote their most probable corrections.

Also, for convenience in writing, put

$$
\begin{array}{ll}
m_1'' = M_1'' - M_1' - A' & m_2'' = M_2'' - M_2' - A' \\
m_1''' = M_1''' - M_1' - B' & m_2''' = M_2''' - M_2' - B' \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}
$$

The normal equations now become

$$
\begin{array}{llll}
[p_1]x_1 & & +p_1''(A)+p_1'''(B)+\ldots & =[p_1m_1] \\
& [p_2]x_2 & +p_2''(A)+p_2'''(B)+\ldots & =[p_2m_2] \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot \\
p_1''x_1+p_2''x_2+\ldots & +[p''](A) & & =[p''m''] \\
p_1'''x_1+p_2'''x_2+\ldots & & +[p'''](B) & =[p'''m'''] \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot
\end{array}
$$

The quantities $x_1, x_2, \ldots$ being merely auxiliary quantities, we eliminate them by substituting their values as found from the first group of normal equations in the second group. We have then

$$\left\{[p'] - \frac{p_1'}{[p_1]}p_1' - \frac{p_2'}{[p_2]}p_2' - \ldots\right\}(A) + \left\{-\frac{p_1''}{[p_1]}p_1' - \frac{p_2''}{[p_2]}p_2' \ldots\right\}(B)$$
$$+ \ldots = [p'm'] - \frac{p_1'}{[p_1]}[p_1m_1] - \frac{p_2'}{[p_2]}[p_2m_2]$$

$$\left\{-\frac{p_1''}{[p_1]}p_1' - \frac{p_2''}{[p_2]}p_2' - \ldots\right\}(A) + \left\{[p''] - \frac{p_1''}{[p_1]}p_1'' - \frac{p_2''}{[p_2]}p_2'' \ldots\right\}(B)$$
$$+ \ldots = [p''m''] - \frac{p_1''}{[p_1]}[p_1m_1] - \frac{p_2''}{[p_2]}[p_2m_2] \quad .$$

. . . .

which may be solved as usual.

These equations may be written

$$[\underline{aa}](A) + [\underline{ab}](B) + \ldots = [al]$$
$$[\underline{ab}](A) + [\underline{bb}](B) + \ldots = [bl]$$

. . . .

where $[\underline{aa}]$, $[\underline{ab}]$, . . . are to be looked on as mere symbols.

In the cases that occur in ordinary work the computation may be still farther shortened. If we arrange the observations in groups containing readings on the same series of signals, then, these readings being of equal value, we have for the first group

$$p_1' = p_1'' = \ldots = {}_1p \text{ suppose}$$
$$p_2' = p_2'' = \ldots = {}_2p \quad \text{"}$$

. . . .

and therefore

$$[p_1] = n_1'\,{}_1p \qquad [p_2] = n_2'\,{}_2p \quad \ldots$$

n_1' being the number of signals sighted at in this group. Similarly for the other groups.

Hence if n_a', n_a'', . . . denote the number of arcs in the several groups, the coefficients of the normal equations become

$$[\underline{aa}] = [p''] - \frac{n_a'}{n_s'}\,{}_1p'' - \frac{n_a''}{n_s''}\,{}_2p'' - \ldots$$

$$[\underline{ab}] = -\frac{n_a'}{n_s'}\,{}_1p''\,{}_1p''' - \frac{n_a''}{n_s''}\,{}_2p''\,{}_2p''' - \ldots$$

$$[\underline{al}] = [p''m''] - \frac{[m_1]}{n_s'}\,{}_1p'' - \frac{[m_2]}{n_s''}\,{}_2p'' - \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where ${}_1p'$, ${}_1p''$, . . . denote any of the equal weights in the several columns of the first group, and ${}_2p'$, ${}_2p''$, . . .; ${}_3p'$, ${}_3p''$, . . .; . . . denote corresponding quantities in the second, third, . . . groups. These weights for the signals sighted at may be taken to be each equal to unity, and, for the signals not sighted at, zero.

After having found the quantities m', m'', . . . by taking the differences between the approximate values of the angles and the several measured values, it is convenient to arrange the formation of the normal equations according to the following scheme:

No. of Group.	$\frac{n_a}{n_s}$	p	$p''m''$	$p'''m'''$	. . .	sum	$\frac{\text{sum}}{n_s}$
1							
2							
3							
. . .							
		$[p]$	$[p''m'']$	$[p'''m''']$	. . .	$[pm]$	

The coefficients of the normal equations may now be written down at sight.

Special Case.—If every arc is full, and every direction is equally well measured in every arc, then

$$|p'| = |p''| = \ldots = n_a$$
$$|p_1| = |p_2| = \ldots = n_s$$

and the normal equations become

$$\left(n_a - \frac{n_a}{n_s}\right)(A) - \frac{n_a}{n_s}(B) - \ldots = |m''| - \frac{|m|}{n_s}$$

$$-\frac{n_a}{n_s}(A) + \left(n_a - \frac{n_a}{n_s}\right)(B) \ldots = |m'''| - \frac{|m|}{n_s}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

By addition, the number of arcs being $n_s - 1$,

$$\frac{n_a}{n_s}\Big((A) + (B) + \ldots\Big) = \frac{|m|}{n_s}$$

Hence

$$n_a(A) = |m''|$$
$$n_a(B) = |m'''|$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

as is evident *à priori*.

148. *Checks of the Normal Equations.*

1. The sum

$$\begin{aligned}[aa] + [ab] + [ac] &+ \ldots \\ + [bb] + [bc] &+ \ldots \\ + [cc] &+ \ldots \\ &+ \ldots\end{aligned}$$

is equal to half the number of observations, less half the number of arcs.

For substituting for $[aa]$, $[ab]$, . . . their values we find the above sum equal to

$$
\begin{aligned}
&[p''] + [p'''] + \dots \\
&\qquad - \frac{p_1''^2 + p_1'''^2 + p_1''p_1''' + \dots}{[p_1]} - \frac{p_2''^2 + p_2'''^2 + p_2''p_2''' + \dots}{[p_2]} - \dots \\
&= \tfrac{1}{2}\{[p'] + [p''] + [p'''] + \dots\} \\
&\qquad + \frac{-p_1' + p_1'' + p_1''' + \dots}{2} - \frac{p_1''^2 + p_1'''^2 + p_1''p_1''' + \dots}{[p_1]} \\
&\qquad + \frac{-p_2'' + p_2'' + p_2''' + \dots}{2} - \frac{p_2''^2 + p_2'''^2 + p_2''p_2''' + \dots}{[p_2]} \\
&\qquad + \dots \\
&= \tfrac{1}{2}\{[p'] + [p''] + [p'''] + \dots\} - \frac{[p_1^2]}{2[p_1]} - \frac{[p_2^2]}{2[p_2]} - \dots
\end{aligned}
$$

Now, $p_1' = p_2' = \dots = 1$, and therefore

$[p'] + [p''] + [p'''] + \dots$ = the number of observations

$\dfrac{[p_1^2]}{[p_1]} + \dfrac{[p_2^2]}{[p_2]} + \dfrac{[p_3^2]}{[p_3]} + \dots$ = the number of arcs.

2. The sum

$$[al] + [bl] + [cl] + \dots = [wl]$$

where $[wl]$ is formed in the same way as $[al]$, $[bl]$, . . .

For

$$
\begin{aligned}
&[al] + [bl] + \dots \\
&\qquad = [p''m''] + [p'''m'''] + \dots \\
&\qquad\quad + [p_1m_1]\left(\frac{p_1'}{[p_1]} - 1\right) + [p_2m_2]\left(\frac{p_2'}{[p_2]} - 1\right) + \dots \\
&\qquad = \frac{p_1'}{[p_1]}[p_1m_1] + \frac{p_2'}{[p_2]}[p_2m_2] + \dots
\end{aligned}
$$

since

$$
\begin{aligned}
&[p''m''] + [p'''m'''] + \dots \\
&\qquad - [p_1m_1] - [p_2m_2] - \dots = [p'm'] \\
&\qquad = 0
\end{aligned}
$$

which proves the proposition.

149. *The Precision of the Adjusted Values.* The first step is to find μ, the m. s. e. of a single direction.

We have generally

$$\mu^2 = \frac{[pvv]}{\text{No. of obs. quan.} - \text{No. of indep. unknowns.}}$$

Now, from the observation equations, Art. 147, it is evident that the number of independent unknowns is equal to the number of arcs n_a, together with the number of signals n_s sighted at, less one. Hence

$$\mu^2 = \frac{[pvv]}{n - (n_a + n_s - 1)}$$

n being the number of directions.

To compute the value of $[pvv]$. Eliminate $x_1, x_2, \ldots$ from the normal equations, Art. 147, and we have the reduced normal equations

$$\begin{array}{llll}
[p_1]x' & + p_1''(A) + p_1'''(B) + \ldots & = & [p_1m_1] \\
 & [p_2]x'' + p_2''(A) + p_2'''(B) + \ldots & = & [p_2m_2] \\
\cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \cdot \\
 & [aa](A) + [ab](B) + \ldots & = & [al] \\
 & \qquad [bb.1](B) + \ldots & = & [bl.1] \\
 & \cdot \quad \cdot \quad \cdot & & \cdot
\end{array}$$

Hence, as in Art. 100,

$[pvv]$

$$= [pmm] - \frac{[p_1m_1]^2}{[p_1]} - \frac{[p_2m_2]^2}{[p_2]} - \ldots - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \ldots$$

$$= [pmm] - \frac{[p_1m_1]^2}{[p_1]} - \frac{[p_2m_2]^2}{[p_2]} - \ldots - [al](A) - [bl](B) - \ldots$$

The quantity $[pmm]$ should always, if possible, be found from the original observations. It will in general be quite different if found from the means forming the different groups of arcs taken as single observations. (See Ex., Art. 62.)

Taking the weight of each observed direction to be o the same value unity, the expressions for $[vv]$ may be written

$$[vv] = [mm] - \frac{[m_1]^2}{n_s'} - \frac{[m_2]^2}{n_s''} - \ldots - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \ldots$$

$$= [mm] - \frac{[m_1]^2}{n_s'} - \frac{[m_2]^2}{n_s''} - \ldots - [al](A) - [bl](B) - \ldots$$

150. **The General Adjustment.**—The general adjustment is carried out as in the case of independent angles. The angle and side equations are formed as in Arts. 117–139, and the solution effected according to the programme of Art. 113.

151. *Ex.* At station Clark Mt., in the triangulation of the Blue Ridge, Va., readings were made with a non-repeating theodolite in the method of arcs. The following, taken from these readings, will be sufficient to illustrate the method of reduction.

The original observations are arranged in sets containing readings on the same groups of signals, and the quantities given in the table below are the remainders found by subtracting the reading of the first direction in each series from the readings of the other directions; that is, $M_1'' - M_1'$, $M_1''' - M_1'$, . . .

At Clark Mt.

Spear.	Humpback	Fork.
00° 00′ 00″.00	24° 09′ 35″.70	78° 26′ 08″.55
.00	39″.40	09″.60
.00	36″.23	09″.33
.00	33″.55	10″.45
00° 00′ 00″.00		78° 26′ 10″.20
.00		11″.03
00° 00′ 00″.00	24° 09′ 36″.10	
.00	37″.40	
.00	36″.53	
.00	38″.15	
.00	39″.00	
	00° 00′ 00″.00	54° 16′ 31″.85
	.00	31″ 94
	.00	32″.03
	.00	36″.19

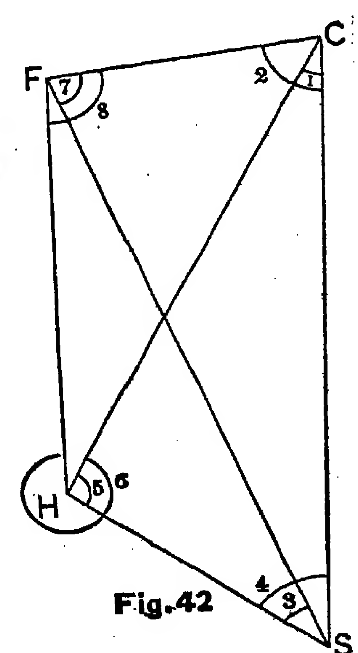


Fig.42

The Local Adjustment.

Assume the most probable values of the angles

Spear,	00° 00′ 00″.00
Humpback,	24° 09′ 36″.90 + (A)
Fork,	78° 26′ 9″.90 + (B)

Take the differences between the approximate values of the angles and the several measured values. We then have

$p'm'$	$p''m''$	$p'''m'''$	Sums.
0.90	− 1.20	− 1.35	
$n_a' = 4$ .00	$_1p'' = 1$ + 2.50	$_1p''' = 1$ − 0.30	
.00	− 0.67	− 0.57	
$n_s' = 3$ .00	− 3.35	+ 0.55	
0.00	− 2.72	− 1.67	− 4.39
$n_a'' = 2$ 0.00		+ 0.30	
$n_s'' = 2$ .00	$_2p'' = 0$	$_2p''' = 1$ + 1.13	
0.00		+ 1.43	+ 1.43
0.00	− 0.80		
.00	+ 0.50		
.00	− 0.37		
.00	+ 1.25		
.00	+ 2.10		
0.00	+ 2.68		+ 2.68
	0.00	− 1.15	
	.00	− 1.06	
	.00	− 0.97	
	.00	+ 3.13	
	0.00	− 0.05	− 0.05
$[p'm'] = 0.00$	$[p''m''] = -0.04$	$[p'''m'''] = -0.29$	$[pm] = -0.33$
$[p'] = 11$	$[p''] = 13$	$[p'''] = 10$	

Next form the table

No. of group.	$\frac{n_a}{n_s}$	n_a	$p'm'$	$p''m''$	$p'''m'''$	Sum.	$\frac{\text{Sum}}{n_s}$
1	$\frac{4}{3}$	4	0	-2.72	-1.67	-4.39	-1.463
2	$\frac{2}{2}$	2	0		$+1.43$	$+1.43$	$+0.715$
3	$\frac{5}{2}$	5	0	$+2.68$		$+2.68$	$+1.340$
4	$\frac{4}{2}$	4	0	.00	-0.05	-0.05	-0.025
Sums,		15		-0.04	-0.29	-0.33	Check

The coefficients of the normal equations

$$[aa] = 13 - \tfrac{4}{3} - \tfrac{5}{2} - \tfrac{4}{2} = +7\tfrac{1}{6}$$
$$[ab] = \quad -\tfrac{4}{3} - \tfrac{4}{2} \quad = -3\tfrac{1}{3}$$
$$[bb] = 10 - \tfrac{4}{3} - \tfrac{2}{2} - \tfrac{4}{2} = +5\tfrac{2}{3}$$
$$[al] = -0.04 + 1.463 - 1.34 + 0.025 = +0.108$$
$$[bl] = -0.29 + 1.463 - 0.715 + 0.025 = +0.483$$
$$[wl] = -1.463 + 0.715 + 1.34 \qquad = +0.592$$

Check (1) $[aa] + [ab] + [bb] = 9\tfrac{1}{2}$

$$= \tfrac{1}{2}(34 - 15)$$
$$= \tfrac{1}{2}(n - n_a)$$

as it should.

(2) $$[al] + [bl] = 0.591$$
$$= [wl]$$

as it should.

The normal equations are

$$7\tfrac{1}{6}(A) - 3\tfrac{1}{3}(B) = +0.108 = [al]$$
$$-3\tfrac{1}{3}(A) + 5\tfrac{2}{3}(B) = +0.483 = [bl]$$

The general solution of these equations gives

$$(A) = +0.1921\,[al] + 0.1130\,[bl]$$
$$(B) = +0.1130\,[al] + 0.2429\,[bl]$$

Substituting for $[al]$, $[bl]$ their values, there result

$$(A) = +0''.075$$
$$(B) = +0''.130$$

and hence the local directions

Spear	0°	00′	00″.000
Humpback,	24°	09′	36″.975
Fork,	78°	26′	10″.030

To find the m. s. e. of a single observation.

The value of $[vv]$, computed according to Art. 150, is found to be 25.7. Therefore

$$\mu = \sqrt{\frac{25.7}{34 - (15 + 3 + 1)}}$$
$$= 1''.2$$

the divisor in this case being 15.

This completes the local adjustment at this station. Proceed similarly at the remaining three stations.

The General Adjustment.

At Clark.

Most probable directions.

Spear,	0°	00′	00″.000
Humpback,	24°	09′	36″.975 + (1)
Fork,	78°	26′	10″.030 + (2)

Weight Equations.

$$(1) = +0.1921 \boxed{1} + 0.1130 \boxed{2}$$
$$(2) = +0.1130 \boxed{1} + 0.2429 \boxed{2}$$

$[vv] = 25.7$ Divisor, 17

At Spear.

Humpback,	0°	00′	00″.000
Fork,	32°	08′	11″.793 + (3)
Clark,	54°	06′	29″.197 + (4)

Weight Equations.

$$(3) = +0.2061 \boxed{3} + 0.0485 \boxed{4}$$
$$(4) = +0.0485 \boxed{3} + 0.1879 \boxed{4}$$

$[vv] = 58.7$ Divisor, 17

At Humpback.

Clark,	0°	00'	00".000
Spear,	101°	44'	3".123 + (5)
Fork,	332°	58'	11".157 + (6)

Weight Equations.

$$(5) = +0.1333\,\boxed{5} + 0.0667\,\boxed{6}$$
$$(6) = +0.0667\,\boxed{5} + 0.1833\,\boxed{6}$$

$[vv] = 106.0$ Divisor, 23

At Fork.

Clark,	0°	00'	00".000
Spear,	79°	35'	42".479 + (7)
Humpback,	98°	41'	43".926 + (8)

Weight Equations.

$$(7) = +0.2970\,\boxed{7} + 0.1304\,\boxed{8}$$
$$(8) = +0.1304\,\boxed{7} + 0.1879\,\boxed{8}$$

$[vv] = 47.5$ Divisor, 17

The angle and side equations are formed as already explained. The angle equations from the triangles SFC ($\epsilon = 10''.773$), HFC ($\epsilon = 7''.386$), SHC ($\epsilon = 9''.789$), and the side equation from the quadrilateral $CSHF$ (pole at C), will be found to be

$$(2) - (3) + (4) + (7) = 0.860$$
$$-(1) + (2) - (6) + (8) = 1.562$$
$$(1) + (4) + (5) = 0.494$$

$$2.609(3) - 1.847(4) + 0.2187(5) - 2.0635(6) + 0.0193(7) + 0.1611(8) = 0.0424$$

From this point the solution is carried through exactly as in Art. 140. The finally adjusted directions will be found to be

At Clark,	Spear,	0° 00' 00".000	At Humpback,	Clark,	0° 00' 00".000
	Humpback,	24° 09' 36".844		Spear,	101° 44' 03".279
	Fork,	78° 26' 10".478		Fork,	332° 58' 10".784
At Spear,	Humpback,	0° 00' 00".000	At Fork,	Clark,	0° 00' 00".000
	Fork,	32° 08' 11".799		Spear,	79° 35' 42".428
	Clark,	54° 06' 29".666		Humpback,	98° 41' 44".536

The m. s. e. of an observation of weight unity is 1".77.
The m. s. e. of the adjusted value of the angle $CSF = 0''.46$.

The form of reduction used on the U. S. Coast Survey for the adjustment of the primary triangulation is essentially the same as that just explained, so far as finding the values of the local corrections is concerned. The method of weighting employed in the general adjustment is not given, as it is not so satisfactory as that in the text. It will be found in *Report*, 1864, app. 14.

152. **Approximate Method of Reduction.**—A very convenient method of approximation may be derived from the normal equations, Art. 147. It depends on the theorem of Art. 115, and hence, if the process is repeated often enough, leads to the same result as the rigorous solution.

In the second group of normal equations, Art. 147, assume

$$X_1 = M_1', \quad X_2 = M_2', \dots$$

then we have as approximate values of A, B, . . .

$$A' = \frac{[p''(M'' - M')]}{[p'']}, \qquad B' = \frac{[p'''(M''' - M')]}{[p''']}, \dots$$

Let x_1 denote the correction to X_1, x_2 to X_2, . . . Then the values A', B', . . . of A, B, . . . substituted in the first group of normal equations, give as approximate values of x_1, x_2, . . .

$$x_1' = \frac{p_1''(M_1'' - M_1' - A') + p_1'''(M_1''' - M_1' - B') + \dots}{[p_1]}$$

$$x_2' = \frac{p_2''(M_2'' - M_2' - A') + p_2'''(M_2''' - M_2' - B') + \dots}{[p_2]}$$

. . . .

which values substituted in the second group give as second approximations to the values of A, B, . . .

$$A'' = \frac{p_1''(M_1'' - M_1' - x_1') + p_2''(M_2'' - M_2' - x_2') + \dots}{[p'']}$$

$$B'' = \frac{p_1'''(M_1''' - M_1' - x_1') + p_2'''(M_2''' - M_2' - x_2') + \dots}{[p''']}$$

. . . .

and so on.

This approximate form of reduction was used on the Ordnance Trigonometrical Survey of Great Britain in the reduction of the principal triangulation. The approximations were carried out only as far as A'', B'', . . . This is in general sufficient in good work.

Instead of finding A', B', . . . as above, it is often more convenient to follow the Ordnance Survey plan, and, instead of deducting M_1', M_2'', . . . from the readings of the signals in the different arcs in order, to add to the readings of the signals in the different arcs quantities which will make the column M' constant throughout. We should then have

$$A' + M' = \frac{[p''M'']}{[p'']}, \quad B' + M' = \frac{[p'''M''']}{[p''']}, \ldots$$

The corrections x and the other approximations are made as before.

Should the observation of the zero direction be wanting in any arc—as, say, the third—the quantity to be added to each reading in this arc is given approximately by the value of X_3 in the first set of normal equations. Thus, given the readings prepared as explained above,

$$\begin{array}{rl} M', & M_1'', \ M_1''' \\ M', & M_2'', \ M_2''' \\ & 0, \quad M_3''' - M_3'' \end{array}$$

to find $M_3'' - M_3'$.

If A', B' have been found from the first two arcs, then from the normal equations

$$M_3' = X_3 = \frac{p_3'M_3' + p_3''(M_3'' - A') + p_3'''(M_3''' - B')}{[p_3]}$$

and

$$M_3'' - M_3' = \frac{p_3''A' - p_3'''(M_3''' - M_3'' - B')}{p_3'' + p_3'''}$$

$$= A' \text{ approx.}$$

Hence the complete form would be

$$\begin{array}{lll} M', M_1'', & & M_1''' \\ M', M_2'', & & M_2''' \\ & A' + M', & M_2''' - M_2'' + A' + M' \end{array}$$

or

$$\begin{array}{lll} 0, & M_1'' - M', & M_1''' - M' \\ 0, & M_2'' - M', & M_2''' - M' \\ A', & & M_2''' - M_2'' + A' \end{array}$$

Ex.

At Clark Mt.

p		$M'' - M'$	$M''' - M'$
4	00° 00′ 00″.00	24° 09′ 36″.22	78° 26′ 09″.48
2	.00		10″.62
5	.00	37″.44	
4		00° 00′ 00″	54° 16′ 32″.99
		A' = 24° 09′ 36″.90	
4	00° 00′ 00″.00	24° 09′ 36″.22	78° 26′ 09″.48
2	.00		10″.62
5	.00	37″.44	
4		36″.90	9″.89
Means		36″.90	9″.88

Corrections to the arcs.

		$M'' - M' - A'$	$M''' - M' - B'$	x
	0″.00	− 0″.68	− 0″.40	− 0″.36
	.00		+ 0″.74	+ 0″.37
	.00	+ 0″.54		+ 0″.27
		0″.00	+ 0″.01	+ 0″.00
		$M'' - M' - x$	$M''' - M' - x$	
4	0″.36	24° 09′ 36″.58	78° 26′ 09″.84	
2	59″.63		10″.25	
5	59″.73	37″.17		
4		36″.90	9″.89	
Means,	59″.94	36″.90	9″.94	
Final values,	00″.00	36″.96	9″.98	

Modified Rigorous Solution.

153. The forms that have been given in the preceding articles for the rigorous adjustment of a triangulation, though analytically very elegant, are somewhat complicated. A method which shall give a marked diminution of work in the reduction without increasing the field work materially is a desideratum.

In the reduction of a long net of triangulation we have seen that labor is saved by breaking the work into two parts, first adjusting at each of the stations for the local conditions, and then using this work in the further adjustment arising from the angle and side equations. Now, if the measurements were made on a uniform plan the local adjustment would be simplified, as we should have a similar problem to solve at each station. This would lead to a great saving of labor, since, if the measurements are made at hap-hazard, the local adjustment may be quite complicated.

If we decide, then, that the observer must work in accordance with some regular form, our next inquiry is, What shall that form be? First, shall the angles be measured independently or in arcs? The point to be aimed at in this as in all work of precision is to get rid of systematic error. The accidental errors are trifling in comparison. When we consider twist of triangulation station from the action of the sun's rays; the influence on distinctness of vision for the same focus for different lengths of lines sighted over; the interruptions that may occur in the course of reading a long arc; the more uniform light that may always be had when the number of signals in use at one time is small, etc., we cannot but conclude that greater precision is to be attained by measuring the angles independently. The errors are more likely to mutually balance. Even Andræ, the author of the most important contributions to the method of directions since Bessel, and who used this method in the triangulation of Denmark, acknowledges

that "in place of observations of directions in arcs it is preferable to return to the old method of Gauss in measuring angles." *

As regards the cost, it must be acknowledged that for an equal number of results, leaving quality out of account, the method of arcs has the advantage. Nowadays, however, when facilities exist for measuring angles by night as well as by day, there is less delay in waiting for suitable conditions than when day work alone had to be depended on.† Taking this into account, the difference in cost would not be great in any case, more especially as a triangulation party is never a very large one.

Having decided that angles should be measured independently, it is in accordance with general experience that instead of spending all of the time of observation in measuring the single angles themselves better results would be obtained by spending part of it in measuring combinations of the angles. A simple form that at once suggests itself would be to close the horizon at each station; that is, to measure all of the angles AOB, BOC, . . . LOA in order round the horizon (see Fig. 20). The local corrections would each be $\frac{1}{n}$ of the discrepancy of the sum of the angles from 360° (Art. 121), and the reduction is thus simple and uniform. However, as measuring the closing angle LOA is the same as measuring the sum angle AOL, it would seem that if we measure one sum angle we ought to measure all possible sum angles.‡ Though the form of adjustment for this combination of measures is a special case of that already given, I shall, at the risk of a little repetition, sketch it in full.

* *Verhandlungen der europäischen Gradmessung*, 1878, p. 47.

† See *C. S. Report* 1880, App. No. 8. Experiments made at Sugar Loaf Mountain, Georgia, have shown that an apparatus cheap and easily operated can be used; that night observations are a little more accurate than those by day, and that the average time of observing in clear weather can be more than doubled by observing at night.

‡ This form of combination was introduced by Gauss in the triangulation of Hanover. A similar form is employed on the New York State Survey. See also *C. S. Report* 1876, App. 20.

154. **The Local Adjustment (Angles).**—Let O be the station occupied, and 1, 2, 3, 4 the stations sighted at in order of azimuth; then the angles to be measured would be

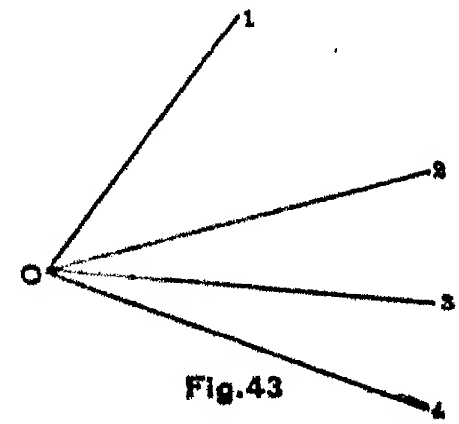


Fig. 43

$$\begin{array}{lll} 102 & & \\ 103 & 203 & \\ 104 & 204 & 304 \end{array}$$

Take the first three as independent unknowns, and let A, B, C denote their most probable values. Also let l_{12}, l_{13}, . . . denote the several measured values.

The observation equations are

$$\left.\begin{array}{rl} +A \qquad\qquad\quad -l_{12} &= v_{12} \\ +B \qquad\quad -l_{13} &= v_{13} \\ +C-l_{14} &= v_{14} \\ -A+B \qquad\quad -l_{23} &= v_{23} \\ -A \qquad\quad +C-l_{24} &= v_{24} \\ -B+C-l_{34} &= v_{34} \end{array}\right\} \qquad (1)$$

and the normal equations

$$\begin{aligned} 3A - B - C &= l_{12} - l_{23} - l_{24} \\ -A + 3B - C &= l_{13} + l_{23} - l_{34} \\ -A - B + 3C &= l_{14} + l_{24} + l_{34} \end{aligned} \qquad (2)$$

Solve these equations, and

$$\begin{aligned} 4A &= 2l_{12} + (l_{13} - l_{23}) + (l_{14} - l_{24}) \\ 4B &= (l_{12} + l_{23}) + 2l_{13} + (l_{14} - l_{34}) \\ 4C &= (l_{12} + l_{24}) + (l_{13} + l_{34}) + 2l_{14} \end{aligned} \qquad (3)$$

It is useful to notice as a check that

$$A + B + C = l_{12} + l_{13} + l_{14} \qquad (4)$$

For practical computation arrange in tabular form as follows. Sum first the horizontal rows, next form the sum

differences, and then place these quantities in the proper columns. Add each column and divide the sums by 4:

l_{12}	l_{13}	l_{14}	Sum_1	
	l_{23}	l_{24}	Sum_2	$\text{Sum}_1-\text{Sum}_2$
		l_{34}	Sum_3	$\text{Sum}_1-\text{Sum}_3$
$\text{Sum}_1-\text{Sum}_2$	$\text{Sum}_1-\text{Sum}_3$	Sum_1		
$4A$	$4B$	$4C$		
A	B	C	Check	

Check Solution.—The form of the expressions found as the values of A, B, C suggests a method of finding these values in accordance with the fundamental principle of the mean. For, writing the value of A in the form

$$A=\frac{2l_{12}+(l_{13}-l_{23})+(l_{14}-l_{24})}{2+1+1} \tag{5}$$

we see that it is the weighted mean of the values

$$\begin{array}{cc} l_{12} & \text{weight } 1 \\ l_{13}-l_{23} & \text{''} \quad \tfrac{1}{2} \\ l_{14}-l_{24} & \text{''} \quad \tfrac{1}{2} \end{array}$$

that is, the value of A is found by taking the weighted mean of the measured value of A and of all the values of A that can be found by combining two other measures.

Similarly for the remaining angles.

The Precision.—From Eq. 3 or from Eq. 5 it is evident that each of the angles A, B, C has the weight 2. The weight P of the angle $-A+B$ is given by (see Art. 101)

$$\begin{aligned}\frac{1}{P}&=[\alpha\alpha]-2[\alpha\beta]+[\beta\beta]\\&=\tfrac{2}{4}-2\tfrac{1}{4}+\tfrac{2}{4}\\&=\tfrac{2}{4}\end{aligned}$$

and $P=2$

The same result will be found for the remaining angles. Hence after the station adjustment each angle has the same weight, being double the weight of a single measured angle in our case of 4 stations sighted at. With n stations sighted at the weight of an adjusted angle would be $\frac{n}{2}$ times the weight of a single measured angle.

Ex. 1. Given at station Oswego, in the triangulation of Lake Ontario (Fig. 38),

$$
\begin{aligned}
l_{12} &= 80^\circ\ 29'\ 46''.10 \\
l_{13} &= 107^\circ\ 19'\ 03''.28 \\
l_{14} &= 138^\circ\ 12'\ 49''.44 \\
l_{23} &= 26^\circ\ 49'\ 16''.61 \\
l_{24} &= 57^\circ\ 43'\ 01''.96 \\
l_{34} &= 30^\circ\ 53'\ 42''.88
\end{aligned}
$$

required the adjusted values.

Solution.

46″.10	63″.28 16″.61	109″.44 61″.96 42″.88	218″.82 78″.57 42″.88	140″.25 175″.94
140″.25	175″.94	218″.82		
186″.35 46″.59 *A*	255″.83 63″.96 *B*	433″.10 108″.28 *C*	218″.83	check.

and the adjusted angles

$$
\begin{aligned}
&80^\circ\ 29'\ 46''.59 \\
&107^\circ\ 19'\ 03''.96 \\
&138^\circ\ 12'\ 48''.28 \\
&26^\circ\ 49'\ 17''.37 \\
&57^\circ\ 43'\ 01''.69 \\
&30^\circ\ 53'\ 44''.32
\end{aligned}
$$

Check Solution.

	Angle *A*.		Angle *B*.		Angle *C*.	
	46″.10	wt. 1	3″.28	wt. 1	49″.44	wt. 1
	46″.67	" $\frac{1}{2}$	2″.71	" $\frac{1}{2}$	46″.16	" $\frac{1}{2}$
	47″.48	" $\frac{1}{2}$	6″.56	" $\frac{1}{2}$	48″.06	" $\frac{1}{2}$
Mean,	46″.59		3″.96		48″.28	

Similarly for the remaining angles.

The weight of each adjusted angle is twice that of each measured angle.

Ex. 2. The angles at station Vanderlip (Art. 142) may be adjusted in the same way as the above.

155. **The General Adjustment (Angles).**—The angle and side equations are formed as explained in Arts. 117–139. The connection between the local and general adjustment is through the weight equations. In our example of 4 stations, in which every angle between every two directions is measured, the weight equations are in three groups, of which the first is (see Eq. 2, p. 333)

$$\begin{aligned} 3[\alpha\alpha] - [\alpha\beta] - [\alpha\gamma] &= 1 \\ -[\alpha\alpha] + 3[\alpha\beta] - [\alpha\gamma] &= 0 \\ -[\alpha\alpha] - [\alpha\beta] + 3[\alpha\gamma] &= 0 \end{aligned}$$

which equations are moderately simple in form.

If we had simply closed the horizon the first group of weight equations would have been

$$\begin{aligned} 2[\alpha\alpha] + [\alpha\beta] + [\alpha\gamma] &= 1 \\ [\alpha\alpha] + 2[\alpha\beta] + [\alpha\gamma] &= 0 \\ [\alpha\alpha] + [\alpha\beta] + 2[\alpha\gamma] &= 0 \end{aligned}$$

which are as complicated as the preceding.

Hence, in a net in which the angles at each station are measured in either of the ways indicated, the solution would be simplified so far as the local adjustment is concerned, in that the adjustment at each station is of a fixed form, but in the general adjustment arising from the angle and side equations the gain is comparatively little.

If, however, instead of finding the corrections to the angles we had found the corrections to the directions of the arms of the angles, the weight equations become much simplified, and therefore also the whole reduction. This idea, which, I believe, is due to Hansen, will now be developed.*

156. **The Local Adjustment (Directions).**—If X, A, B, C denote the readings of the 4 directions 1, 2, 3, 4 from O, then since $A - X$, $B - X$, $C - X$ correspond to the

* See *Die preussische Landestriangulation*. Berlin, 1874, seq.

A, B, C in the angle adjustment, the observation equations may be written

$$
\begin{array}{rrrrl}
-X & +A & & & -l_{12}=v'_{12}\\
-X & & +B & & -l_{13}=v'_{13}\\
-X & & & +C & -l_{14}=v'_{14}\\
 & -A & +B & & -l_{23}=v'_{23}\\
 & -A & & +C & -l_{24}=v'_{24}\\
 & & -B & +C & -l_{34}=v'_{34}
\end{array}
\qquad (1)
$$

and the normal equations

$$
\begin{array}{rrrrl}
3X & -A & -B & -C & =-l_{12}-l_{13}-l_{14}\\
-X & +3A & -B & -C & =l_{12}-l_{23}-l_{24}\\
-X & -A & +3B & -C & =l_{13}+l_{23}-l_{34}\\
-X & -A & -B & +3C & =l_{14}+l_{24}+l_{34}
\end{array}
\qquad (2)
$$

Adding these equations, there results

$$0=0$$

and therefore the unknowns cannot be found without some further relation connecting them. The reason of the indeterminate form is that directions are nothing but the angles which the rays $O1$, $O2$, . . . make with some common zero ray whose position is not fixed, and which may therefore be taken arbitrarily. To carry out the solution it will be most convenient to fix the zero ray by making the arbitrary assumption

$$X+A+B+C=0 \qquad (3)$$

By adding this to each of the normal equations they reduce to

$$
\begin{array}{rl}
4X = & -l_{12}-l_{13}-l_{14}\\
4A = & l_{12}-l_{23}-l_{24}\\
4B = & l_{13}+l_{23}-l_{34}\\
4C = & l_{14}+l_{24}+l_{34}
\end{array}
\qquad (4)
$$

which give the values of X, A, B, C directly.

The computation may be rendered quite mechanical by arranging in tabular form:

1	2	3	4	Sums.
	l_{12}	l_{13} l_{23}	l_{14} l_{24} l_{34}	Sum_1 Sum_2 Sum_3
$-\text{Sum}_1$	$-\text{Sum}_2$	$-\text{Sum}_3$		
$4X$ X	$4A$ A	$4B$ B	$4C$ C	Check

The transformation of the normal equations (2) into (4) by means of the arbitrary relation (3) is allowable. For since the sum of the coefficients of the unknowns in each of equations 2 is zero, whatever values of X, A, B, C satisfy those equations $X+a$, $A+a$, $B+a$, $C+a$, where a is any constant, will also satisfy them. Hence whatever set of values is taken to satisfy the equations, the differences $A-X$, $B-X$, $C-X$ will be the same in value. Therefore by arbitrarily fixing the zero direction we find determinate values for the corrections to the other directions.

Ex. Take that of Ex. 1, Art. 154. The tabular scheme, writing down the seconds only, is

	46″.10	63″.28 16″.61	109″.44 61″.96 42″.88	218″.82 78″.57 42″.88
− 218″.82	− 78″.57	− 42″.88		
− 218″.82 − 54″.70 X	− 32″.47 − 8″.12 A	37″.01 9″.25 B	214″.28 53″.57 C	

and $A-X$, $B-X$, $C-X$ give the same values of the angles as found before. Hence, whether we find the local corrections to the angles or to the directions

of the arms of the angles, the result is the same. One method or the other may, therefore, be used, as is most convenient.

To avoid the use of large numbers certain approximate values may be assumed for the directions, and the corrections to these approximate values found. (Compare Art. 81.) Thus if (X), (A), (B), (C) denote the corrections to assumed values of X, A, B, C, we may proceed as follows:

Assumed approximate angles

Sodus,	0°	00′	00″ + (X)
Vanderlip,	80°	29′	46″ + (A)
Duck,	107°	19′	03″ + (B)
Stony Pt.,	138°	12′	49″ + (C)

Tabular Form.

1	2	3	4	Sums.
	+ 0″.10	+ 0″.28	+ 0″.44	+ 0″.82
		− 0″.39	− 1″.04	− 1″.43
			− 3″.12	− 3″.12
− 0″.82	+ 1″.43	+ 3″.12		
− 0″.82	+ 1″.53	+ 3″.01	− 3″.76	
− 0″.20	+ 0″.38	+ 0″.75	− 0″.94	
(X)	(A)	(B)	(C)	

and the adjusted angles are as before.

157. **The General Adjustment (Directions).**—The form of the normal equations 4, Art. 156, shows that at each station the quantities X, A, . . . are determined independently of one another. The weight of each direction at a station, as shown by the weight equations, is represented by the number of stations sighted at. If, therefore, (1), (2), . . . denote the corrections to the values X, A, . . . locally adjusted, which arise from the angle and side equations, we may consider X, A, . . . as independent quantities, all of equal weight and subject to certain rigorous conditions, and proceed to carry out the solution according to the simple form of Art. 110. Hence, as the solution breaks into two simple problems of adjusting quantities as if independently observed, we see the advantage of adjusting the directions instead of the angles.

Ex. The quadrilateral Buchanan, Brulè, Aminicon, Lester, in the triangulation of Lake Superior.

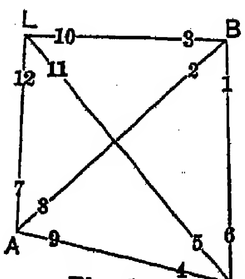


Fig. 44

	Direction.	Measured Angle.	Local Adj. Angle.
At Buchanan,	2–1	47° 57′ 36″.25	36″.21
	3–1	97° 26′ 41″.29	41″.32
	3–2	49° 29′ 05″.15	5″.11
At Brulè,	5–4	40° 25′ 47″.49	47″.36
	6–4	71° 58′ 27″.31	27″.45
	6–5	31° 32′ 40″.22	40″.09
At Aminicon,	8–7	37° 47′ 05″.51	5″.17
	9–7	97° 51′ 03″.11	3″.45
	9–8	60° 03′ 58″.62	58″.28
At Lester,	11–10	51° 00′ 39″.77	40″.00
	12–10	92° 43′ 49″.42	49″.20
	12–11	41° 43′ 08″.97	9″.20

The local adjustment is carried through as in Art. 154, it being more convenient to adjust the angles directly rather than the directions.

Taking the locally adjusted angles as independent, we next find the corrections to the *directions* arising from the angle and side equations.

The angle equations, formed in the usual way from the triangles Buchanan, Brulè, Aminicon ($\varepsilon = 1''.37$); Brulè, Aminicon, Lester ($\varepsilon = 1''.19$); Lester, Buchanan, Brulè ($\varepsilon = 1''.19$); and the side equation, from the quadrilateral itself (pole at Lester), are

$$-(1)+(2)-(4)+(6)-(8)+(9)=-0''.57$$
$$-(1)+(3)-(5)+(6)-(10)+(11)=-0''.22$$
$$-(4)+(5)-(7)+(9)-(11)+(12)=+1''.18$$

$$-0.14(1)-0.90(2)+1.04(3)+1.24(4)-2.95(5)$$
$$+1.72(6)+1.50(7)-1.36(8)-0.14(9)=-0''.80$$

The number of stations pointed at from each station occupied being 3, the weight of each locally adjusted angle is the same throughout the net.

Hence we have the correlate normal equations

I.	II.	III.	IV.	
$+6.0$	$+2.0$	$+2.0$	$+0.94$	$=-0.57$
$+2.0$	$+6.0$	-2.0	$+5.84$	$=-0.22$
$+2.0$	-2.0	$+6.0$	-5.84	$=+1.18$
$+0.94$	$+5.84$	-5.84	$+19.22$	$=-0.80$

$\therefore$ I. $=-0.273$ III. $=+0.377$

II. $=+0.136$ IV. $=+0.045$

and the corrections

$(1) = +0''.13$	$(7) = -0''.31$
$(2) = -0''.31$	$(8) = +0''.21$
$(3) = +0''.18$	$(9) = +0''.10$
$(4) = -0''.05$	$(10) = -0''.14$
$(5) = +0''.11$	$(11) = -0''.24$
$(6) = -0''.06$	$(12) = +0''.38$

On the Breaking of a Net of Triangulation into Sections for Convenience of Solution.

158. In a long chain of triangulation or in a complicated net the simultaneous solution of the condition equations would be very troublesome, not from any principle involved, but from its very unwieldiness. Accordingly it is necessary to break the work into sections and solve each section by itself. As this breaking into sections causes more or less disturbance of the local conditions at the lines of breaking off, each section should be as large as can be conveniently managed, and the lines of breaking off should be so chosen as to disturb as few conditions as possible. By the method of adjustment explained in Arts. 153–157 much larger sections can be taken than by any other. This is a strong argument in favor of its use.

The contradictions that occur at the lines of breaking off of the several sections are most conveniently bridged over by means of the principle of Art. 115. As an example we may cite the reduction of the principal triangulation of the British Ordnance Survey. 'There were 920 condition equations to be satisfied in the net. The following was the method of solution employed:*

"The triangulation was divided into a number of parts or figures, each affording a not unmanageable number of equations of condition. One of these being corrected or computed independently of all the rest, the corrections so obtained were substituted (so far as they entered) in the

* *Account of the Principal Triangulation*, p. 272.

equations of condition of the next figure, and the sum of the squares of the remaining corrections in that figure made a minimum. The corrections thus obtained for the second figure were substituted in the third, and so on."

In the triangulation of Mecklenburg* this method was carried out even more systematically. The adjustment was divided into five groups of 22, 22, 22, 21, 22 condition equations respectively. The corrections resulting from the solution of group I. were carried, so far as they entered, into group II., and this group solved, and so on through groups III., IV., V. The whole operation was repeated four times, and the small contradictions still remaining were distributed empirically.

For an interesting conference on the whole question see *Comptes Rendus de l'Association Géodésique Internationale*, 1877.

Adjustment of a Triangulation for Closure of Circuit.

159. In the adjustment of a triangulation we have so far considered it only with reference to a single measured base. We have seen how at each station the discrepancy arising from sum angles and from closure of the horizon can be got rid of, and also how in a net joining several stations the conditions arising from closure of triangles and from equality of lengths of sides computed by different routes can be satisfied. There remains the question as to the mode of procedure when several bases enter whose lengths are known and whose positions have been fixed astronomically. Special cases would be where a circuit of triangulation closed on the initial line, and where a secondary system is to be made to conform to two lines in a primary system, the primary lines being assumed to be known in length and position.

The conditions to be satisfied in the adjustment are four in number—that the value of a base computed from another

* *Grossherzoglich mecklenburgische Landesvermessung.* Schwerin, 1882.

should agree with the measured value in azimuth, in length, and in latitude and longitude of one of the end points.

The measured angles of the triangles connecting the bases having been already adjusted with reference to one base for the local and general conditions, the additional corrections necessary to satisfy the closure of the circuit will be small. But little difference in the results will therefore be found by making a simultaneous solution of all of the condition equations of closure and a solution by successive approximation according to the method of Art. 115.

We shall consider only a single chain of triangles, all tie lines of the system being rejected. This on account of simplicity, and also for the reason stated above, that, in good work the corrections being small, it gives results practically close enough—nearly the same, in fact, as a rigorous solution. If thought necessary in any special case a rigorous solution of the condition equations can be carried out by the method of correlates.

160. **Adjustment for Discrepancy in Azimuth.**—Let 1–2 and 5–6 be two bases connected through intermediate stations 3, 4 by a single chain of the best-shaped triangles that can be selected from the net.

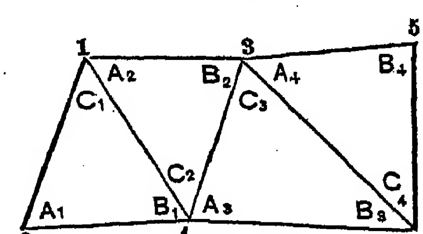


Fig. 45

In computing the base 5–6 from 1–2 the sides 1–4, 3–4, 3–6 are at once sides of continuation and bases, according to the triangles considered. For example, in the triangle 1 2 4 the side 1–4 is a side of continuation from the base 1–2, but in 1 3 4 the side 1–4 is a base with reference to 3–4 as a side of continuation.

In the chain of triangles let

$A_1, A_2, \ldots$ be the angles opposite to the sides of continuation.

$B_1, B_2, \ldots$ the angles opposite to the bases in order of computation.

$C_1, C_2, \ldots$ the angles opposite to the flank sides.

Let z, z_1 denote the values of the measured azimuth of the bases 1–2, 5–6 respectively. These values are assumed to be correct, and receive no change in the adjustment.

A geodetic computation* of the azimuth of the base 5–6 from the base 1–2 is now made, using the values of the angles of the intervening triangles resulting from the adjustment for local and general conditions. Call the value of the azimuth of 5–6 computed in this way z'.

Now, reckoning azimuth in the usual way from the south as origin, and the direction of increase from south to west, it is easily seen, by passing from 1–2 along the sides 1–4, 4–3, 3–6, that the excess l_z of the observed value z_1 over the computed value z' of 5–6 is given by

$$-(C_1)+(C_2)-(C_3)+(C_4)=l_z \qquad (1)$$

where (C_1), (C_2), . . . denote the corrections to the angles C_1, C_2, . . . This is the azimuth condition equation.

In order that the correction arising from the azimuth equation may not disturb the conditions of closure existing among the angles of the triangles, it is necessary that the corrections to the angles should satisfy the conditions

$$\begin{aligned}(A_1)+(B_1)+(C_1)&=0\\(A_2)+(B_2)+(C_2)&=0\\ \cdot\quad\cdot\quad\cdot\quad&\cdot\quad\cdot\end{aligned} \qquad (2)$$

The unknowns in equations 1, 2 are subject to the relation

$$(A_1)^2+(B_1)^2+\ldots=\text{a min.}$$

Call k_1, k_2, k_3, k_4 the correlates of equations (2), and k the correlate of equation (1), and we have the correlate equations

$$\begin{array}{llll}k_1 =(A_1) & k_2 =(A_2) & k_3 =(A_3) & k_4 =(A_4)\\ k_1 =(B_1) & k_2 =(B_2) & k_3 =(B_3) & k_4 =(B_4)\\ k_1-k=(C_1) & k_2+k=(C_2) & k_3-k=(C_3) & k_4+k=(C_4)\end{array}$$

* For methods of doing this see Lee's *Tables and Formulæ*, Washington, 1873; *Coast Survey Report*, 1875.

whence the normal equations

$$
\begin{array}{rrrrr}
3k_1 & & & & -\ k = 0 \\
& 3k_2 & & & +\ k = 0 \\
& & 3k_3 & & -\ k = 0 \\
& & & 3k_4 & +\ k = 0 \\
-k_1 & +k_2 & -k_3 & +k_4 & +4k = l_z
\end{array}
$$

If we had n triangles instead of four, the normal equations would be of similar form, and, solving, we should find

$$k = \frac{3l_z}{2n}$$

$$+k_1 = -k_2 = +k_3 = \ldots = \frac{l_z}{2n}$$

and therefore the corrections

$$(A_1) = \frac{1}{2n}l_z, \quad (A_2) = -\frac{1}{2n}l_z, \ldots$$

$$(B_1) = \frac{1}{2n}l_z, \quad (B_2) = -\frac{1}{2n}l_z, \ldots$$

$$(C_1) = -\frac{1}{n}l_z, \quad (C_2) = +\frac{1}{n}l_z, \ldots$$

Hence the rule: *Divide the excess of the observed over the computed azimuth by the number of triangles, and apply one-half of this quantity to each of the angles adjacent to the flank sides on one side of the chain, and the total quantity, with the sign changed, to the third angle. The signs are reversed for the angles on the other flank.*

If the azimuth mark is not on a triangulation side the line of azimuth may be swung on to such a side by adding the angle between the mark and the side.

Ex. The sketch represents the secondary triangulation of Long Island Sound. The sides 1–2, 14–15 are the lines of junction with the primary sys-

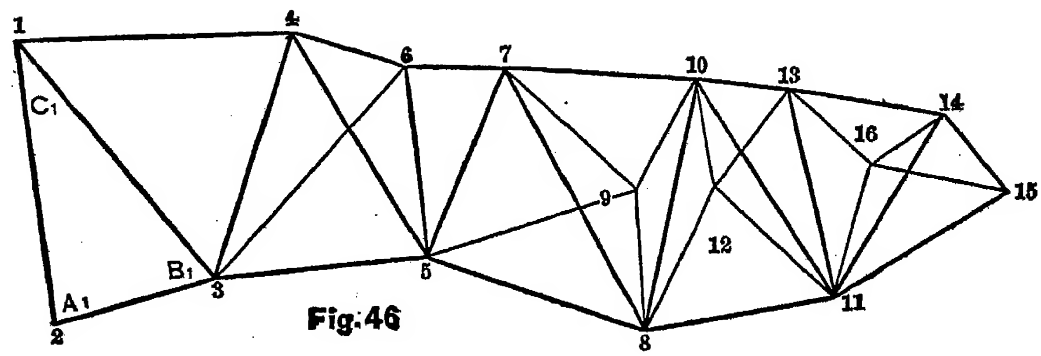


Fig. 46

tem. These lines are assumed to remain unchanged in azimuth in the adjustment, and the secondary system is to be made to conform to them.

The main chain of triangles joining the two primary lines is indicated in the figure by heavy lines. The number of these triangles is 11. The system has been adjusted for local and geometrical conditions, and the resulting angles of these triangles are as follows:

	Angle.	log sin diff. 1″	Sph. excess.		Angle.	log sin diff. 1″	Sph. excess.
A_1	82° 49′ 19″.25	0.27		A_7	58° 25′ 08″.08	1.29	
B_1	64° 55′ 40″.32	0.99	3″.66	B_7	82° 16′ 16″.16	0.29	3″.63
C_1	32° 15′ 04″.09			C_7	39° 18′ 39″.39		
A_2	52° 37′ 47″.98	1.61		A_8	69° 12′ 06″.02	0.80	
B_2	74° 10′ 50″.56	0.60	4″.96	B_8	71° 58′ 54″.18	0.68	3″.03
C_2	53° 11′ 26″.42			C_8	38° 49′ 02″.83		
A_3	69° 20′ 16″.03	0.79		A_9	59° 18′ 26″.33	1.25	
B_3	67° 43′ 43″.33	0.86	3″.53	B_9	102° 08′ 26″.94	—0.45	1″.33
C_3	42° 56′ 04″.16			C_9	18° 33′ 08″,05		
A_4	39° 31′ 29″.66	2.55		A_{10}	67° 38′ 52″.77	0.87	
B_4	120° 36′ 36″.11	—1.25	1″.31	B_{10}	70° 26′ 23″.26	0.75	2″.41
C_4	19° 51′ 55″.54			C_{10}	41° 54′ 46″.38		
A_5	83° 19′ 31″.08	0.25		A_{11}	29° 43′ 55″.99	3.69	
B_5	68° 57′ 47″.75	0.81	1″.42	B_{11}	55° 23′ 09″.36	1.45	2″.13
C_5	27° 42′ 42″.58			C_{11}	94° 52′ 56″.78		
A_6	87° 25′ 26″.66	0.09					
B_6	44° 10′ 45″.25	2.17	3″.47				
C_6	48° 23′ 51″.56						

A geodetic computation for latitude, longitude, and azimuth was carried through from the line 1–2 to 14–15, using the above angles. It was found that approximately

$$\text{observed az. of } 14\text{–}15 - \text{computed az. of do.} = -2''.93$$

Hence the corrections to the angles of the triangles for this discrepancy in azimuth are for the

$$\text{first triangle } (A_1) = -0''.13 \quad \text{second triangle } (A_2) = +0''.13$$
$$(B_1) = -0''.13 \qquad (B_2) = +0''.13$$
$$(C_1) = +0''.26 \qquad (C_2) = -0''.26$$

and so on.

161. **Adjustment for Discrepancy in Bases.**—This is fully explained in Arts. 168–170.

Using the last form given in Art. 170, we may write the base-line equation

$$[\delta_A(A) - \delta_B(B)] = l$$

from which, using the values of A_1, B_1, . . . found in the azimuth adjustment as first approximations, further corrections to the angles are found, as in equations 5, Art. 170. Since the angles C do not enter into this adjustment, the corrections resulting will not disturb the adjustment for azimuth already made.

The advantage of this method of proceeding is that the subsequent work does not disturb any adjustment already made and thus render it necessary to make a new approximation. The labor is reduced to a minimum, and the results obtained are practically close enough.

For an example see Ex. 1, Art. 170.

162. **Adjustment for Discrepancy in Latitude and Longitude.**—The corrections to the angles arising from discrepancy in azimuth and in bases having been applied, the value of one base computed from another will agree with the measured value in direction and length.

The discrepancy in position, as shown by the differences between observed and computed latitudes and longitudes, alone remains. This discrepancy, being small, may be eliminated closely enough by distributing it proportionally from one end of the chain of triangles to the other, according to Bowditch's rule as given in Ex. 5, Art. 110—that is, the error in latitude in proportion to the longitudes, and the error in longitude in proportion to the latitudes of the several stations. Each station, being thus made slightly eccentric, is next reduced to centre, when the whole net will be consistent.

CHAPTER VII.

APPLICATION TO BASE-LINE MEASUREMENTS.

163. During the present century two forms of apparatus have been used in the measurement of primary bases, the compensation bars and the metallic-thermometer apparatus. On the English Ordnance Survey the two principal lines, the Lough Foyle and Salisbury Plain bases, were measured with the Colby compensation bars. Most of the bases of the U. S. Coast Survey and five of the eight bases of the U. S. Lake Survey were measured with the Bache-Würdemann compensation apparatus. On the Continent of Europe the Bessel metallic-thermometer apparatus is very generally used.

Indications are not wanting that both forms will be supplanted before long by an apparatus consisting of simply a single metallic bar.*

The essential part of any form of base apparatus consists of one or two bars of metal, usually from 4 to 6 metres in length and of about 40 × 15 mm. cross-section. If the extreme points of a bar are the limits of measure, so that a measurement is made by contact [end-measures], two bars are necessary. If, however, the length of a bar is considered to lie between two marks made on it [line-measures], so that in a measurement the transition from one bar to the next depends, not on the stability of the bar, but on some outside appliance, only one bar is necessary.

Descriptions of the various forms of apparatus used on different surveys will be found in reports of those surveys.

* See Ibanez, *Zeitschr. für Instrumentenkunde*, 1881. Also Art. 166.

164. **Precision of a Base-Line Measurement.**—For clearness it will be necessary to outline the principles on which the measurement is made.

First, we must find the length of the measuring bar in terms of some standard of length; and as the measurements of the line itself are made at various temperatures, the coefficients of expansion of the metals in the measuring apparatus must also be known. Comparisons must, therefore, be made with the standard during wide ranges of temperature; and as these comparisons are fallible, the results found for length and expansion will be more or less erroneous.

The principle involved in the measurement is exactly the same as in common chaining with chain and pins. There are, indeed, various contrivances for getting a precision not looked for in chaining, such as for aligning the measuring bar, for finding the inclination of each position of the bar, and for establishing fixed points for stopping at and starting from in measurement. But these make no change in the essential principle.

The errors in the value of a base line may, therefore, be considered to arise from two principal sources, comparisons and measurement. Experience has shown that the main error arises from the comparisons, and that, even if our modes of measuring the base line itself were perfect, the precision of the final value would be but little increased so long as the methods of comparison are in their present state. Thus in the Lake Survey primary bases, if the field work had been without error, the total p. e. of the bases would have been diminished only about $\frac{1}{45}$ part.

These errors differ essentially in character. An error arising from the comparisons, being the same for each bar measurement, is cumulative for the whole base, while errors arising in the measurement of the base itself, were the measurements repeated often enough and the conditions sufficiently varied, would tend to mutually balance, and could, therefore, be treated by the strict principles of least

squares. But as the number of measurements is not often more than 2 or 3, and as these are made usually at about the same season of the year, only a comparatively rough estimate of the precision is to be looked for.

As a check on the field work a base is usually divided into sections by setting stones firmly in the ground at approximately equal intervals along the line, so that instead of being able to compare results at the end points only, we may compare results just as well at 6 or 8 points. In this way a better idea of the precision of the work is obtained, as we have 6 or 8 short bases to deal with instead of a single long one.

We proceed now with the problem of determining the precision of measurement. It may be stated as follows: A base is measured in n sections with a bar of a certain length, each section being measured n_1 times. By the first measurement the first section contains M_1' bars, the second M_1'' bars, . . . ; by the second measurement the first section contains M_2' bars, the second M_2'' bars, . . . ; and so on. The weights of the measurements in order being p_1', p_2', . . . ; p_1'', p_2'', . . . ; . . . respectively, required the m. s. e. of the most probable value of the base.

Let $V_1 =$ most prob. value of first section

$V_2 =$ most prob. value of second section

. . . .

then we have the observation equations

First section, $V_1 - M_1' = v_1'$ wt. p_1'

$V_1 - M_1'' = v_1''$ wt. p_1''

. . . . .

Second section, $V_2 - M_2' = v_2'$ wt. p_2'

$V_2 - M_2'' = v_2''$ wt. p_2''

. . .

and so on.

Now, either of two assumptions may be made.

(a) In the first place, that the precision of the measurement of each bar is the same throughout the different sections.

We have, then, nn_1 equations containing n unknowns, and the normal equations are

$$\begin{aligned} +[p_1]V_1 \qquad\qquad &= [p_1M_1] \\ +[p_2]V_2 &= [p_2M_2] \\ \cdot \quad \cdot \quad \cdot \end{aligned}$$

whence $V_1, V_2, \ldots$ are known, and therefore the whole line $V = V_1 + V_2 + \ldots + V_n$ is known.

The mean-square error μ of an observation of weight unity—that is, of a single measurement of a bar—is given by (see Art. 99)

$$\mu = \sqrt{\frac{[pvv]}{\text{No. of obs.} - \text{No. of indep. unknowns.}}}$$

$$= \sqrt{\frac{[pvv]}{n(n_1 - 1)}}$$

Now, the length of the measuring bar being taken as the unit of measurement, the weight of a section, as depending on the measurement, may be expressed in terms of the number of bars measured.

For since μ is the m. s. e. of a measurement of a single bar, the m. s. e. of the measurement of a length of M bars is $\mu\sqrt{M}$. Hence $\frac{1}{M}$ is the weight of a measurement of length M when the weight of a measurement of the unit of length is unity.

Writing, therefore, for the weights p their values in terms of M,

$$\mu = \sqrt{\frac{1}{n(n_1 - 1)}\left[\frac{vv}{M}\right]}$$

In the case usually occurring in practice, where the line is measured twice, we may put this formula in a form more

convenient for computation. For if the first measurement of the n_1 sections gave lengths $M_1, M_2, \ldots$, and the second measurements gave lengths $M_1 + d_1$, $M_2 + d_2, \ldots$ for the same sections in order, then, since

$$[vv] = \left[\left(+\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2\right] = \frac{[dd]}{2}$$

we have for the m. s. e. of one measurement of a bar and for the mean of two measurements respectively

$$\sqrt{\frac{1}{2n}\left[\frac{dd}{M}\right]} \text{ and } \frac{1}{2}\sqrt{\frac{1}{n}\left[\frac{dd}{M}\right]}$$

Hence the m. s. e. of a single measurement and of the mean of the two values of the whole base are

$$\sqrt{\frac{[M]}{2n}\left[\frac{dd}{M}\right]} \text{ and } \frac{1}{2}\sqrt{\frac{[M]}{n}\left[\frac{dd}{M}\right]}$$

the number of bars in the line being $[M]$.

Ex. The Bonn Base, measured in 1847, near Bonn, Germany, with the original Bessel metallic-thermometer apparatus. The base was a broken one, the two parts making an angle of 179° 23′. Each part was measured twice, as follows:*

	Differences.	No. of bars.
	L	
Northern Part, Sec. 1	− 0.183	116
Sec. 2	+ 0.094	87
Sec. 3	− 0.013	61
		—— 264
Southern Part, Sec. 1	− 0.097	92
Sec. 2	+ 0.095	60
Sec. 3	+ 0.757	131
		—— 283

* *Das rheinische Dreiecksnetz.* Berlin, 1876.

Hence the m. s. e. of the northern part, arising from errors of measurement only, is

$$\tfrac{1}{3}\sqrt{\frac{264}{3}\left(\frac{.183^2}{116}+\frac{.094^2}{87}+\frac{.013^2}{61}\right)}=\pm\overset{L}{0.093}$$

and the m. s. e. of the southern part is

$$\tfrac{1}{3}\sqrt{\frac{283}{3}\left(\frac{.007^2}{92}+\frac{.095^2}{60}+\frac{.757^2}{131}\right)}=\pm\overset{L}{0.327}$$

The other two main sources of error are:

1. Error in comparison of the measuring bars with one another
2. Error in the determination of their length.

The m. s. e. arising from these sources are respectively

$\pm\overset{L}{0.386}$, $\pm\overset{L}{0.313}$ for the northern part

± 0.391, ± 0.335 for the southern part

Remembering that these latter errors are systematic, we have, finally,

$$\begin{aligned}\text{m. s. e. of base} &= \sqrt{.093^2+.327^2+(.386+.391)^2+(.313+.335)^2}\\ &= \overset{L}{1.07}\end{aligned}$$

(b) In the second place, if we assume that the law of precision of the measurements of the different sections is unknown, and that these sections are independent, we have for the mean of the values of the several sections and their m. s. e.

$$V_1=\frac{[p_1M_1]}{[p_1]}$$

$$\mu_{V_1}{}^2=\frac{[p_1v_1v_1]}{[p_1](n_1-1)}=\frac{[v_1v_1]}{n_1(n_1-1)}\ \text{since}\ p_1'=p_1''=\ldots=\frac{1}{M_1}$$

$$V_2=\frac{[p_2M_2]}{[p_2]}$$

$$\mu_{V_2}{}^2=\frac{[p_2v_2v_2]}{[p_2](n_1-1)}=\frac{[v_2v_2]}{n_1(n_1-1)}\ \text{since}\ p_2'=p_2''=\ldots=\frac{1}{M_2}$$

. . . .

If V denotes the whole line, so that

$$V = V_1 + V_2 + \ldots + V_n$$

then, since the measurements are independent,

$$\begin{aligned} \mu_V^2 &= \mu_{V_1}^2 + \mu_{V_2}^2 + \ldots \\ &= \frac{1}{n_1(n_1 - 1)}\left([v_1v_1] + [v_2v_2] + \ldots\right) \end{aligned}$$

and the (m. s. e.)^2 of a single measurement of the line

$$\frac{1}{n_1 - 1}\left([v_1v_1] + [v_2v_2] + \ldots\right)$$

The number of bars in the line being $[M]$, we have for the average value of the (m. s. e.)^2 of a single measurement of a bar

$$\frac{1}{n_1 - 1} \frac{[v_1v_1] + [v_2v_2] + \ldots}{[M]}$$

If, for example, the line has been measured twice, and $d_1, d_2, \ldots d_n$ are the differences of the measurements of the several sections, then

$$[v_1v_1] = \frac{d_1^2}{2}$$

$$[v_2v_2] = \frac{d_2^2}{2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

and therefore

$$\mu_V^2 = \frac{[d^2]}{4}$$

and the (m. s. e.)^2 of a single measurement of a bar is $\frac{1}{2}\frac{[d^2]}{[M]}$.

Ex. The Chicago Base, measured in 1877 with the Repsold metallic-thermometer apparatus belonging to the U. S. Engineers. The base was divided into 8 sections and was measured twice.

Section.	No. of Bars.	Diff. of Measures.
I.	227.25	$\overset{mm}{-1.3}$
II.	230.25	+ 2.5
III.	234.50	+ 2.3
IV.	232.	+ 0.7
V.	231.	+ 1.5
VI.	225.	+ 1.1
VII.	300.50	+ 1.3
VIII.	196.80	− 0.2

Taking the errors of the different sections as independent, the p. e. of the mean of the two measures of the base is

$$0.6745 \sqrt{\frac{[d^2]}{4}} = \overset{mm}{1.46}$$

The p. e. arising from the other sources of error were

(1) measuring bar,	$\pm \overset{mm}{6.38}$
(2) metallic thermometer,	± 2.82
(3) elevation above mean tide, N. Y.,	± 0.36

Assuming these to be independent, the p. e. of the Chicago Base at sea level is

$$\sqrt{1.46^2 + 6.38^2 + 2.82^2 + 0.36^2} = \overset{mm}{7.14}$$

Of the two assumptions (a) and (b), the first is due to Bessel and the second to Andræ. For a more elaborate discussion of the points involved see *Astron. Nachr.*, Nos. 1924, 1935, and *V. J. Schr. d. Astr. Ges.*, 1878, p. 69.

165. **Length of Base and Number of Bases necessary in a Triangulation.**—In computing a side of a triangle from another and shorter side as base, the loss of precision arising from the influence of the acute angle opposite to the base is the greater the more acute that angle is. To guard against this, if a measured base were a side of a triangle it would be necessary that its length be

about the average length of the triangle sides of the system. In the older work the practice was to measure long bases, even as long as 20 kilometres and over. The modern practice was introduced by Prof. Schwerd, of Speyer, Germany, in 1819. By measuring a short base of 860^m and checking on to a line computed from a base of 15 kilometres, he found a result agreeing within 0^m.1. His conclusion was that "with a small expenditure of time, trouble, and expense the base of a large triangulation can be determined by a small exactly measured line." *

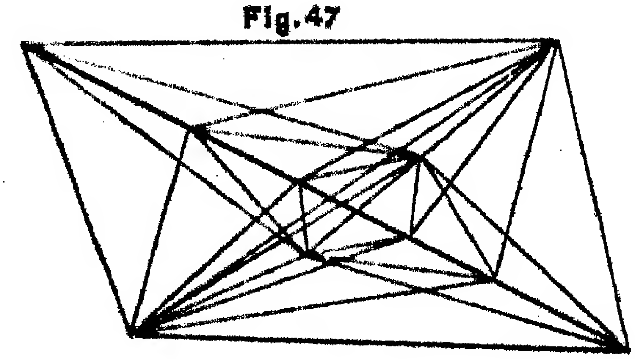
A similar conclusion was reached by Gen. Ibanez as the result of measuring the base of Madridejos, Spain, in 1859. He says: "The question, so much disputed between French and German geometers, as to whether it is necessary to measure long bases, or if short bases are sufficient, had occupied the attention of the observers. Taking advantage, therefore, of the favorable opportunity which presented itself to them, they proposed to compare the results obtained from the direct measure of the whole base with those calculated from a special triangulation depending on the central section of the base." †

The adjustment of this triangulation net gave rise to 36 angle equations and 28 side equations containing 90 unknown quantities. The net is shown in the figure.

"We give in the following table the results of the direct measures reduced to the sea level, and their comparison with the values found trigonometrically:

Measured.	Triangulation.
m	m
3077.459	3077.462
2216.397	2216.399
2766.604	
2723.425	2723.422
3879.000	3879.002
14662.885	14662.889

Fig. 47



* *Die kleine Speyerer Basis.* Speyer, 1822. † *Astron. Nachr.*, No. 1460.

"The remarkable agreement which the two operations present is sufficient authority to limit the length of bases, and to be satisfied with those of 2 or 3 kilometres in length, always on the condition that they are joined to the sides of the main triangulation by means of a system of lines arranged so as to be able to apply to it the proper method of adjustment."

If errors arising from the angles of the triangles are neglected, it is easy to show that if a short base M is measured n times, and the arithmetic mean of the n measurements taken as base from which a line equal to nM is derived by triangulation, the precision of this line is the same as if it had been measured once directly.*

For if $\mu_M =$ the m. s. e. of a length M, then $\mu_M \sqrt{n}$ is the m. s. e. of a length nM. Also, the m. s. e. of the arithmetic mean of n measurements of the line M is $\frac{\mu_M}{\sqrt{n}}$, and when from this line the line nM is derived trigonometrically its m. s. e. is $\frac{\mu_M}{\sqrt{n}} n = \mu_M \sqrt{n}$, agreeing with the preceding.

Since, however, the principal errors arise from the triangulation, the advantage is with the longer base.

We may estimate the relative amount of influence of errors in the base and of errors in the angles in a chain of triangles, on the value of any side computed from the base, as follows:

Assuming the triangles to be approximately equilateral, we have (Ex. 9, Art. 111)

$$\mu_{a_n}^2 = \mu_b^2 + \tfrac{2}{3} \mu^2 \sin^2 1'' \, b^2 n$$

where a_n is the computed side, b the base, n the number of triangles intervening, and μ the m. s. e. of a measured angle.

Suppose, for the sake of fixing our ideas, that $b = 10000^m$, then

$$u_{a_n}^2 = \mu_b^2 + 1600 \, n\mu^2 \text{ millimetres.}$$

* *Gradmessung in Ostpreussen*, p. 36.

Now, with a primary base apparatus a precision of a m. s. e. of 2 mm. in 1000 m. can be easily reached.* Hence from the above formula we see that even in a short chain of triangles the error of a side arising from the error of the base is small in comparison with that arising from the errors in the angles measured.

Also, since the m. s. e. of a side arising from errors in the angles measured increases as the square root of the number of triangles from the base (Ex. 9, Art. 111), we conclude that it is better to measure bases frequently with moderate precision rather than to measure a few at long intervals in the net, but with great precision.

But little is gained in precision by repeating the measurement of the base many times. We have seen in Art. 164 that the main sources of error to be feared arise from the comparisons with the standards and are independent of the measurement of the line. Accordingly, though bases have in the present century been measured from 6 to 8 times, it is the general custom now to do so only twice. In this way any gross error is checked and a sufficiently close precision determination of the measurement can be found. As a compromise between the error arising from the connection with the triangulation and the systematic error introduced by the base apparatus itself, the general practice is to measure bases of from 4 to 8 kilometres† in length, or about one-sixth of the triangle sides. Besides, long bases are not always to be had in positions just where wanted as triangle sides, as the configuration of most countries will not allow of it.

* The p. e. of the Madridejos Base (1858), Spain, is $\frac{1}{3865800}$ part; of the Grossenhain Base 1872), Saxony, $\frac{1}{3000000}$ part; of the Atlanta Base (1872–1873), U. S., $\frac{1}{561880}$ part; of the Chicago Base (1877), U. S., $\frac{1}{1053200}$ part; of the Sandusky Base (1878), U. S., $\frac{1}{1148600}$ part.

† For example, Salisbury Plain (1849), England, 11.1 kil.; Halland (1863), Sweden, 7.3 kil.; Oran (1867), Algeria, 9.4 kil.; Harlem (1868–1869), Holland, 6.0 kil.; Atlanta (1872–1873), U. S., 9.3 kil.; Radautz (1874), Austria, 4.6 kil.; Udine (1874), Italy, 3.2 kil.; Chicago (1877), U. S., 7.5 kil.; Vich (1877), Spain, 2.5 kil.; Göttingen (1880), Germany, 5.2 kil.; Meppen (1883), Germany, 7.0 kil.

166. Measuring a base line is not necessarily a difficult operation. The great trouble with all forms of apparatus hitherto employed arises from temperature changes. But with the apparatus proposed by Mr. E. S. Wheeler, in which the measuring bar is simply a bar of metal packed in melting ice, this difficulty would be altogether overcome. The length of the bar would remain unchanged throughout the measurement, as its temperature is kept constant, being that of melting ice. The same temperature could at any time be had at which to find the length of the bar in terms of the official standard of length.

The apparatus might be constructed as follows: The measuring bar, a bar of steel 25 mm. in diameter and 6 m. in length, placed in a circular cast-steel tube ¼ m. in diameter, made stiff by bracing, but as light as possible. Along the top of this tube slots of about 75 mm. in width would be cut to allow the introduction of ice around the bar. The hole for drainage would be at the centre of the tube on the under side. For supports during the measurement two trestles placed 1¼ m. from the ends would be best. Effects of flexure would be got rid of by having the graduation marks showing the length of the bar placed on the neutral axis of the bar. The reading microscopes, alignment apparatus, sector and level for determining the inclination of the bar during measurement, such as those made by Repsold for the U. S. Engineers. The mode of measurement the same as with the Repsold apparatus.* The amount of computation necessary to reduce the measurements made in this way would be small in comparison with that required with the forms of apparatus at present in use.

167. **Connection of a Base with the Main Triangulation.**—We shall now consider the best method of connecting a base line with a side of the main triangulation with the least possible loss of precision. This, like most

* See *Professional Papers Corps of Engineers* U. S. A., No. 24, for description of the Repsold base apparatus.

geodetic questions, is not to be decided by least-squares' methods alone.

There are two points to be considered—as little loss of precision as possible, combined with economy of work in the measurement and reduction of the intervening triangulation. To secure the first we must use only well shaped triangles, avoiding in particular very acute angles opposite the bases. To secure the second as few stations as possible should be occupied.

The solution is a tentative one. Various forms of connection may be tried, and the m. s. e. of the triangle sides computed from the bases by the methods of Chapter V., on the hypothesis of a regularity of figure which conforms more or less closely to the case in hand. Many of the results will be found in Ex. 10–15, Art. 111. These results we shall now make use of.

Taking the length of a triangulation side to be from 15 to 30 miles, as giving the best results in angle measurements, the proper length of the base would be from 3 to 5 miles—that is, about one-sixth of a triangle side. Now, if the connection between the base AB and a triangle side 6 times the base is through a chain of equilateral triangles of the form of Fig. 12; or of the rhomboidal form of Fig. 15, composed of two similar triangles, in which the first diagonal BB' is 6 times the base; or of the rhomboidal form of Fig. 16, in which the second diagonal is 6 times the base, the respective weights of the derived side are, roughly, as the numbers 16, 3, 7.

Hence, so far as the precision is concerned, the advantage is with the first form. When, however, we consider that this form requires 12 stations to be occupied, while the third needs only 6, we see that when amount of labor as well as degree of precision is taken into account the advantage is with the latter form. The second form, though the most economical, is condemned by the acute angles that occur in it opposite to the base.

Similar comparisons of other forms of connection will

lead us to the conclusion that the rhomboidal form (Fig. 16), in which a side equal in length to about 6 times the base can be found by passing through two sets of similar triangles from the base, is the normal form, and that the forms used in practice should approximate to it as closely as the nature of the country will allow. As examples take the connections of the Coast Survey Atlanta Base (1872–1873), the Lake Survey Chicago Base (1877), and the Prussian Göttingen Base (1880).

Fig. 48

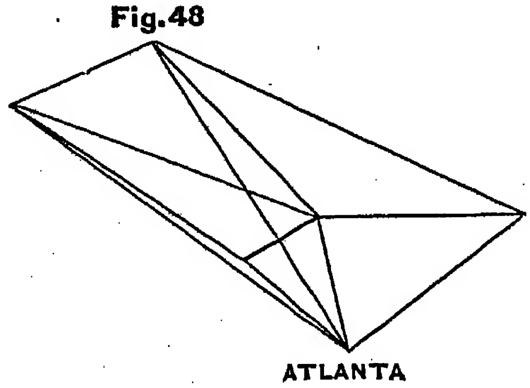


Fig. 49

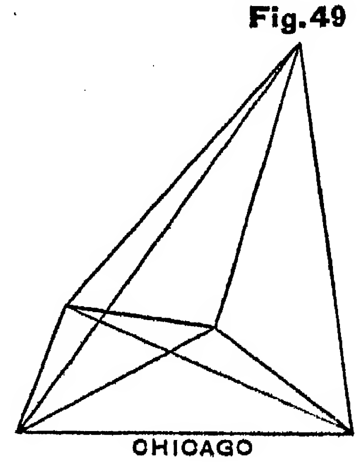
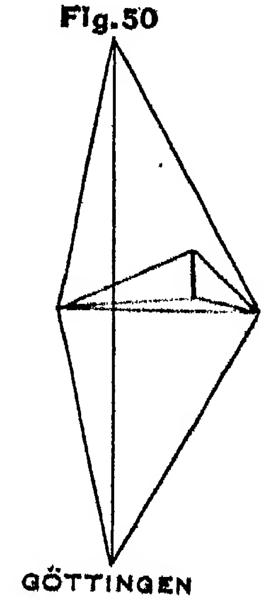


Fig. 50



The form of the triangulation net to connect the base with the main triangulation being decided on, it becomes a question (but only a secondary one) to decide with what care the several angles in this net should be measured in order to attain the greatest precision with a given expenditure of labor. This will be found fully discussed in *Zeitschr. für Vermess.*, 1882, pp. 122 seq.

168. **Adjustment of a Triangulation when more than one Base is considered.**—In all discussions so far we have considered only one measured base. But in a net of triangles, bases must be measured at intervals, for reasons already assigned. In computing an intermediate side from different bases discrepancies will be found. How shall these discrepancies be treated?

The triangulation adjustment could have been made in

such a way that no discrepancy would show itself in passing from one measured base to another. It was only necessary to introduce equations connecting the bases in the form of an ordinary side equation, thus (Fig. 12),

$$\frac{a}{b} = \frac{\sin A_1}{\sin B_1} \frac{\sin A_2}{\sin B_2} \ldots$$

where a, b are two bases, and A_1, B_1, . . . are the angles of continuation.

There is no objection to doing this so long as the bases have been measured with the same apparatus and the intervening triangulation is first-rate. But as the discrepancy between the measured value of the base and its value as computed from another base through the triangulation affords a good test of the quality of the work, it is better not to introduce an equation connecting the bases into the first adjustment.

169. Suppose that the triangulation has been adjusted in sections, each with reference to a single measured base, as explained in Chapter VI., and we wish to adjust for the discrepancy arising from computing one base from another through a chain of the best-shaped triangles in the system. It will be sufficient to confine our attention to this chain of triangles, all tie lines being rejected.

(1) *Rigorous Solution.*—The condition equation to be satisfied, arising from the connection of the bases, may be written

$$\frac{a + (a)}{b + (b)} = \frac{\sin \{A_1 + (A_1)\}}{\sin \{B_1 + (B_1)\}} \frac{\sin \{A_2 + (A_2)\}}{\sin \{B_2 + (B_2)\}} \ldots$$

where a, b, A_1, B_1, . . . are measured values, and (a), (b), (A_1), (B_1), . . . are their most probable corrections.

Taking logs. and reducing to the linear form (Ex. 4, Art. 64),

$$-\delta_a(a) + \delta_b(b) + [\delta_A(A) - \delta_B(B)] = l$$

where l is the excess of the observed over the computed value of log a, and δ_a, δ_b, δ_A, δ_B are the log. differences as usual.

Also, since the angles of each triangle in the chain must satisfy the condition of closure, we have the conditions

$$(A_1)+(B_1)+(C_1)=0$$
$$(A_2)+(B_2)+(C_2)=0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with

$$\frac{(a)^2}{\mu_a^2}+\frac{(b)^2}{\mu_b^2}+\left[\frac{1}{\mu^2}\left\{(A)^2+(B)^2+(C)^2\right\}\right]=\text{a min.}$$

where μ_a, μ_b are the m. s. e. of the bases, and μ_1, μ_2, . . . the m. s. e. of the three angles of each of the triangles in the chain.

The solution may be carried out by the method of correlates.

If the bases have been measured with different apparatus the question of the comparison of the measuring bars with the standards of length is the important one. After that has been satisfactorily settled the connection of the bases may be treated as above.

Ex. This mode of reduction may be illustrated by the simple case of a triangle having two sides and the three angles measured.

In the triangle W. Base, E. Angle, W. Angle (ABC), Sandusky Base, there were measured

$$BC = 6742.420 \text{ ft.} \pm 0.010 \text{ ft.}$$
$$AC = 6602.386 \pm 0.010$$

From the adjustment of the triangulation with reference to one measured base, and which was carried out by the methods already explained in Chapter VI., the three adjusted angles of the triangle were found to be

$$BAC = 1^\circ\ 07'\ 51''.35 \pm 0''.20$$
$$ABC = 1^\circ\ 06'\ 26''.74 \pm 0''.20$$
$$ACB = 177^\circ\ 45'\ 41''.91 \pm 0''.20$$

Required the most probable values of the sides and angles of the triangle from a second adjustment into which the two measured sides enter.

If (a), (b), (A), (B), (C) are the corrections to the above quantities in order, the condition equations are

$$(A) + (B) + (C) = 0$$

$$\frac{6742.420 + (a)}{6602.386 + (b)} = \frac{\sin\{1^\circ\ 07'\ 51''.35 + (A)\}}{\sin\{1^\circ\ 06'\ 26''.74 + (B)\}}$$

or, reducing the latter to the linear form,

$$-0.644(a) + 0.658(b) - 1.089(A) + 1.067(B) = -0.044$$

with

$$\frac{(a)^2}{(.01)^2} + \frac{(b)^2}{(.01)^2} + \frac{(A)^2}{(.20)^2} + \frac{(B)^2}{(.20)^2} + \frac{(C)^2}{(.20)^2} = \text{a min.}$$

that is,

$$400(a)^2 + 400(b)^2 + (A)^2 + (B)^2 + (C)^2 = \text{a min.}$$

The solution of these equations gives

m. $(a) = +\ 0.00003$	$(A) = -\ 0''.02$
$(b) = -\ 0.00003$	$(B) = +\ 0''.02$
	$(C) = \ \ 0''.00$

This example is noteworthy as showing the combination of heterogeneous measures in the same minimum equation.

(2) *Approximate Solutions*.—The two of most importance are (a) when the angles alone are changed, (b) when the bases alone are changed. Either of these is practically more important than the rigid solution, as a base line in good work receives a very small correction from the adjustment.

(a) When the angles alone are changed. The formulas to be used in this case follow from those of the rigorous method just given by putting the base corrections equal to zero. Thus, if all of the adjusted angles are of the same weight, then, since $(a) = 0$, $(b) = 0$, the base line equation becomes

$$[\delta_A(A) - \delta_B(B)] = l \qquad (1)$$

with

$$[(A)^2 + (B)^2 + (C)^2] = \text{a min.}$$

The angle equations are as before.

Hence if k is the correlate of the base-line equation, we have, by eliminating the angle equation correlates,

$$\begin{aligned}
(A_1) &= \quad (2\delta_A' + \delta_B')k \\
(B_1) &= - \ (\delta_A' + 2\delta_B')k \\
(C_1) &= - \ (\delta_A' - \delta_B')k \\
\\
(A_2) &= \quad (2\delta_A'' + \delta_B'')k \\
(B_2) &= - \ (\delta_A'' + 2\delta_B'')k \\
(C_2) &= - \ (\delta_A'' - \delta_B'')k \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{2}$$

whence, by substituting in the base-line equation,

$$k = \frac{l}{2[\delta_A^2 + \delta_A \delta_B + \delta_B^2]}$$

Hence the corrections to the angles are known.

A still further approximation may be made. It is evident that the corrections to the angles C are small compared with those to A and B, and that they vanish when $A = B$. Hence, as we have assumed the triangles to be well shaped, we may take

$$(C_1) = (C_2) = \ldots = 0 \tag{3}$$

The angle equations then become

$$\begin{aligned}
(A_1) + (B_1) &= 0 \\
(A_2) + (B_2) &= 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{4}$$

whence

$$\begin{aligned}
(A_1) &= -(B_1) = \frac{\delta_A' + \delta_B'}{2} k \\
(A_2) &= -(B_2) = \frac{\delta_A'' + \delta_B''}{2} k \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{5}$$

where

$$k = \frac{2l}{[(\delta_A + \delta_B)^2]}$$

Ex. 1. To find the changes in the angles resulting from the equation connecting the lengths of the lines 1–2 and 14–15 in the triangulation of Long Island Sound (Fig. 46).

The excess of the log. of the observed value of 14–15 over the value computed from 1–2 through the triangulation is 4.00 in units of the sixth decimal place.

If the lines themselves receive no correction the condition equation, expressed in the linear form, is (see table, p. 346)

$$0.27(A_1) - 0.99(B_1) + 1.61(A_2) - 0.60(B_2) + \ldots = 4.00$$

Now,

$$[(\delta_A + \delta_B)^2] = 51.48$$

and therefore

$$(A_1) = -(B_1) = \frac{0.27 + 0.99}{51.48} \times 4.00 = 0''.10$$

$$(A_2) = -(B_2) = \frac{1.61 + 0.60}{51.48} \times 4.00 = 0''.17$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

the corrections required.

Ex. 2. To find the precision of a side in a chain of triangles joining two bases.

Having found the adjusted angles, we may find the weight of the log. of any side of continuation as the m^{th} in a chain of n triangles joining two bases, all of the angles being of the same weight.

For simplicity in writing take $m = 2$, $n = 3$.

The condition equations are

$$(A_1) + (B_1) + (C_1) = 0$$
$$(A_2) + (B_2) + (C_2) = 0$$
$$(A_3) + (B_3) + (C_3) = 0$$

$$\delta_A'(A_1) - \delta_B'(B_1) + \delta_A''(A_2) - \delta_B''(B_2) + \delta_A'''(A_3) - \delta_B'''(B_3) = l$$

and the function F, whose weight is to be found,

$$F = \delta_A'(A_1) - \delta_B'(B_1) + \delta_A''(A_2) - \delta_B''(B_2)$$

The solution follows at once from Eq. 15, Art. 111. The result is

$$u_F = \tfrac{2}{3}\left[\delta_A^2 + \delta_A\ \delta_B + \delta_B^2\right]_1^2 \frac{\delta_A'''^2 + \delta_A'''\ \delta_B''' + \delta_B'''^2}{\left[\delta_A^2 + \delta_A\,\delta_B + \delta_B^2\right]_1^3}$$

the summation of the δ's extending between the limits indicated.

As an example of a different method of treating this special case (a), the Coast Survey connection of three primary bases, the Fire Island, Massachusetts, and Epping, may be cited.*

The triangulation net connecting the bases is conceived to consist of three branches, one for each base and proceeding therefrom to a line in common. Adjusting each branch, three independent values of the length and p. e. of the line of junction are obtained. The weighted mean of these values is taken to be the most probable value of the line of junction. This value, as well as the length of each base, is considered to be exact. The adjustment of each branch of the triangulation is then repeated with an additional equation fixing the ratio of the length of the base to the line of junction. In making this adjustment the form of solution given in Art. 90 will be found very convenient in handling the extra equation.

In the final computation of the lengths of the triangle sides "from any one of the measured base lines, we shall, as we recede from it, obtain a proportional amount of the influence of the measure of the other two lines as we approach them, and finally, reaching one or the other, the effect of that base from which we set out is lost. Each distance will have assigned to it its most probable value; we shall have no discrepancy whatever in the geometrical figure of the triangulation, and the resulting sides and angles will have nearly the same probability as those derived from a theoretically perfect solution."

(b) By changing the bases alone. The argument for changing the bases only is that as the computed discrepancy is always small, and as the triangulation has been already adjusted, it is a great saving of labor to change all the triangles proportionally, and thus get a consistent result by a method which, if not rigorous, is good enough. This is the more evident if we consider with what accuracy geodetic work can now be done.

* *Report*, 1865, app. No. 21.

Let $b_1, b_2, \ldots$ be the measured lengths of the bases, and $p_1, p_2, \ldots$ their weights.

If s is the most probable length of a common side computed from the bases, and $\lambda_1, \lambda_2, \ldots$ are the ratios of the lengths of the bases to this side, then

$$s = \frac{b_1}{\lambda_1} \quad \text{weight } p_1\lambda_1^2, \text{ and weight of } \log\frac{b_1}{\lambda_1} \text{ is } p_1b_1^2$$

$$s = \frac{b_2}{\lambda_2} \quad \text{"} \quad p_2\lambda_2^2 \quad \text{"} \quad \log\frac{b_2}{\lambda_2} \text{ is } p_2b_2^2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The weighted mean value of $\log s$ is

$$\frac{p_1b_1^2 \log\frac{b_1}{\lambda_1} + p_2b_2^2 \log\frac{b_2}{\lambda_2} + \ldots}{p_1b_1^2 + p_2b_2^2 + \ldots}$$

With this value of $\log s$, leaving the angles of the triangles as they are, the sides may be recomputed, when, of course, the bases will be changed

We have seen in Art. 164 that in base-line measurements the error arising from errors of comparison of the measuring bars is the main one, and that the error arising from the measurement of the line itself is relatively small. Hence we may take the m. s. e. of a base as proportional to the length, and the weight, therefore, inversely as the square of the length. The weight of each computed value of $\log s$ will then be the same, and we have now

$$\log s \doteq \frac{1}{n}\left[\log\frac{b}{\lambda}\right]$$

As the weights taken are tolerably close, this simple formula will give a result near enough for most work.

It is evident that if we compute each base from all of the others, and take the mean of the logarithmic values found, consistent values of the sides throughout the triangulation will be found by starting from any base.

Thus in the Coast Survey example already referred to,

	Epping Base.	Mass. Base.	Fire Island Base.
Measured,	3.9403143.4	4.2387077.4	4.1479535.3
Computed from Epping,		4.2387115.5	4.1479556.3
" " Mass.,	3.9403075.8		4.1479518.4
" " Fire Id.,	3.9403122.4	4.2387094.6	
Means,	3.9403113.9	4.2387095.8	4.1479536.7

There will now be no contradiction in the values of any triangle side deduced from the different bases.

CHAPTER VIII.

APPLICATION TO LEVELLING.

170. The object of levelling is to find the difference of height of two or more points on the surface of the earth. As only differences of height, and not absolute values, are required, some one height may be selected as the standard of reference, and to it any value at will may be assigned. The common custom in geodetic work is to take what is called the mean level of the ocean as the standard height. To be able to refer to it at any time a fixed mark is established, say at New York, and readings are taken to the surface of the ocean at high and low water for a lunation at least. The height of the fixed mark above mean tide is computed from these readings. Then all that is to be done in order to find the elevation of any point above mean tide is to run a line of levels between this point and the fixed mark just described.

There are two methods of levelling in common use in geodetic work: spirit levelling of precision and trigonometrical levelling. The difference between the two consists mainly in the greater lengths of lines sighted over in the latter case.

Spirit Levelling of Precision.

171. For descriptions of instruments employed see *Nivellement de Précision de la Suisse*, par Hirsch et Plantamour, Genève et Bale, 1867, seq.; *Report Chief of Engineers U. S. A.*, 1880, App. OO; *Report U. S. Coast Survey*, 1879 App. 15, 1880 App. 11; *Professional Papers Corps of Engineers* U. S. A., No. 24, Chap. xxii

The method of observing used on the Coast Survey is as follows: Two levelling rods are used. "Two lines are run simultaneously, with the rods usually at different distances from the instrument; and to prevent the gradual accumulation of error supposed to be due to running constantly in one direction, alternate sections are run in opposite directions. Each station of the instrument, therefore, contains two backsights and two foresights. In stations 1, 3, 5, etc., rod A, backsight, is invariably read first; then rod B, backsight; then rod A, foresight; and finally rod B, foresight. In stations 2, 4, 6, rod B takes precedence of A in both back and fore sights."

An important source of error in spirit levelling, and one very commonly overlooked, is the change in the length of the levelling rod from variations of temperature. From experiments made by the Prussian Land Survey, in which the rods were compared daily with a steel standard, the following fluctuations in length were found for four rods made of seasoned fir:*

Rod 13,	from	May	19	to	Aug. 18,	0.51	mm. per metre.
14,	"	"	20	"	" 15,	0.46	" "
9,	"	"	24	"	Sept. 6,	0.37	" "
10,	"	"	24	"	" 6,	0.43	" "

It is quite possible that errors from this source may largely exceed the errors arising from the levelling itself. Each field party should therefore be provided with the means of making a daily comparison of the rods used with a standard of length. A steel metre and a micrometer microscope mounted on a stand would be all that would be necessary.

It is a curious fact in spirit levelling, but one abundantly verified, that a much greater discrepancy is to be expected in a duplicate line of levels if the lines have been levelled over in opposite directions than if both have been run in the same direction. This was noticed long since in a line

* *Nivellements der Landesaufnahme*, vol. v. Berlin, 1883.

of levels run under the direction of a committee of the British Association from the Bristol Channel to the English Channel (Portishead to Axmouth), a distance of 74 miles. The increase in the discrepancy of the forward and backward levels was continuous. Thus at 12 miles from Portishead it was 0.35 ft., at 23 miles 0.53 ft., at 37 miles 0.70 ft., at 49 miles 0.82 ft., at 59 miles 0.92 ft., and at 74 miles 1.03 ft.*

In the winter of 1878–1879 a line of levels was run, under the direction of the U. S. Engineers, from Austin, Miss., to Friar's Point, Miss., a distance of about 44 kilometres. "All lines were levelled in duplicate and in opposite directions. When using two rods both were used on the same line, rod No. 2 being used for the first, third, . . . backsight and for the second, fourth, . . . foresight, and rod No. 3 for the second, fourth, . . . backsight and for the first, third, . . . foresight, always using rod No. 2 on the closing bench mark." The results for the principal bench marks are given in the following table, where it will be seen that the discrepancy in the forward and backward levels increases from one end of the line to the other: †

Distance.	Bench Mark.	Discrepancy.
	Austin I.	
m 16505	Trotter's Landing.	mm + 16.8
4464	Glendale.	+ 16.8
15950	Delta.	+ 23.6
6602	Friar's Point I.	+ 36.5

Experience on the survey of the great lakes and of the Mississippi River, and also on the survey of India, has shown that the personal peculiarities of the observer enter

* *Report British Association*, 1838. † *Report Chief of Engineers U. S. A.*, 1879, p. 1944.

largely into levelling work. It would seem that a much more reliable result will be obtained if the line between two points whose difference of height is required has been run over not only in opposite directions, but in opposite directions by the same observer. The personal bias is probably due most largely to the fact that, the ends of the bubble not being sharply defined, different observers estimate the positions of these end points differently. To eliminate its effect completely the line should be levelled in opposite directions by each of a large number of observers.

Again, as both rod and bubble should be read simultaneously at each foresight and at each backsight, and as this is physically impossible for one observer, the effect of the unequal heating of the instrument by the sun, even when shaded by an umbrella, has a tendency to cause a change in the position of the bubble in the interval between the readings of the rod and bubble. This error is cumulative, and its reduction to a minimum depends upon the skill of the observer.

The rule adopted by the European Gradmessung for allowable discrepancy in a duplicate line of levels is that the p. e. of the difference in height of two points one kilometre apart should in general not exceed 3 mm., and should in no case exceed 5 mm. On the U. S. Coast Survey, for short distances a discrepancy between two levellings of a distance of D kilometres of an amount not exceeding $5\sqrt{2D}$ mm. is allowed.

172. **Precision of a Line of Levels.**—The precision of a line of levels will be given by the m. s. e. of a single levelling of a unit of distance, which we shall take to be one kilometre. The problem is quite analogous to that already discussed in Art. 164, the unit of distance there being the length of a measuring bar.

If we suppose that in running a single line of levels between two bench marks, A and B, the ground is equally favorable throughout, and that none but accidental errors may be expected to enter, then if μ is the m. s. e. of the

unit of distance (one kilometre), the m. s. e. of a distance of D kilometres would be

$$\mu\sqrt{D}$$

in other words, *the weight of a levelling of a distance of D kilometres would be inversely proportional to that distance.*

Strictly speaking, we should find the weight from the m. s. e. μ_1, μ_2, . . . arising from all sources of error that enter into the work, such as from the comparisons of the levelling rod with the standard, from the nature of the country levelled over, from effects of change of length of sight, etc. Then, considering the errors arising from these sources to be independent, we should find finally the m. s. e. of each line levelled over to be $\sqrt{[\mu^2]}$, and the weight of the line would therefore be as $\frac{1}{[\mu^2]}$.

The uncertainty attendant on estimating these sources of error is so great, and the influence of the distance so largely exceeds the influence of the others, that it is sufficient to adopt the rule first given of weighting as the inverse distance. It follows from this rule that better work is to be looked for if short sights are taken. Even with a first-rate telescope sights should not be taken to exceed 100 metres.* As the rod is more easily read by the observer at short distances, the loss of time from the more frequent settings of the instrument is not so great as would at first appear.

Suppose now that in finding the difference of height of two bench marks, A and B, n, lines of levels have been run, and that the results have been compared at intermediate bench marks in succession D_1, D_2, . . . D_n kilometres apart;

* On the precise levelling of the Coast Survey, "where the slope of the ground is steep the distances may be taken as great as possible, and on comparatively level ground they may range from 50 to 150 metres, according to the condition of the weather and atmosphere."

The U. S. Engineers follow the rule, "The lengths of sight will depend on the condition of the atmosphere, but the rod should always be near enough to be seen distinctly. It will be seldom that lengths of sight greater than 150 metres can be taken."

In the Prussian Land Survey "the length of sight since 1879 has not been taken over 50^m. Only in special cases—for example, in crossing streams—is it permitted to exceed 50^m, whereas a shorter sight is frequently taken."

then if v_1', v_1'', . . . ; v_2', v_2'', . . . ; . . . denote the residual errors of the differences of height in the n sections, and p_1', p_1'', . . . ; p_2', p_2'', . . . ; . . . denote the weights of the observed differences, we have, as in Art. 164—

(1) On the hypothesis that the precision for each unit of distance (one kilometre) is the same throughout the different sections,

$$\mu^2 = \frac{[pvv]}{n(n_1 - 1)}$$

$$= \frac{1}{n(n_1 - 1)}\left[\frac{vv}{D}\right]$$

since the weights are inversely as the distances.

(2) On the hypothesis of the independence of the sections between the intermediate bench marks, the average value of μ is given by

$$\mu^2 = \frac{n_1}{n_1 - 1}\left\{[v_1v_1] + [v_2v_2] + \ldots\right\}\frac{1}{[D]}$$

When the number of levellings is two, and the observed differences of height at the several bench marks D_1, D_2, . . . D_n kilometres apart give discrepancies of d_1, d_2, . . . d_n respectively, then the above formulas reduce to

$$\frac{1}{2n}\left[\frac{d^2}{D}\right] \text{ and } \frac{1}{2}\frac{[d^2]}{[D]}$$

The m. s. e. of a single levelling of the whole line would be respectively

$$\sqrt{\frac{[D]}{2n}\left[\frac{d^2}{D}\right]} \text{ and } \sqrt{\frac{[d^2]}{2}}$$

and for the m. s. e. of the mean of the two measurements these values would each be divided by $\sqrt{2}$.

Ex. 1. In the precise Levelling (1880) of the U. S. Coast Survey on the Mississippi River two lines were run simultaneously between every two bench marks. The following are the results from B.M. LIV. to B.M. LV.

B.M.	Distance.	Rod A.	Rod B.	Difference.
	kil.	*m.*	*m.*	*mm.*
	3.447	− 0.8702	− 0.8713	1.1
	3.806	+ 0.5805	+ 0.5738	6.7
	1.204	− 0.2154	− 0.2204	5.0
	3.038	+ 0.2667	+ 0.2664	0.6

It will be found that $\mu = 2^{mm}$ nearly.

Ex. 2. The distance AB has been levelled n different times. Calling d_2 the difference between the first and second measurements, d_3 the difference between the first and third, and so on, show that the m. s. e. of an observation of weight unity is

$$\sqrt{\frac{n[d^2] - [d]^2}{n(n-1)}}$$

Ex. 3. The difference of level of two points, A and B, is found by two routes whose lengths are D_1, D_2 kil. respectively. If the discrepancy in the results is d, the m. s. e. of one kilometre is $\dfrac{d}{\sqrt{D_1 + D_2}}$,

173. **Adjustment of a Net of Levels.**—If a line of levels is to be run between two points a sufficient check of the accuracy of the work will in general be afforded by running the line over at least twice. Comparisons may be made at intermediate points not too far apart, and if the discrepancies found are within the limits already mentioned the mean of the results may be taken as giving the elevations sought. This method was used by the U. S. Engineers in determining the heights of the great lakes between Canada and the United States above mean tide.

But when a complete topographical survey of a country is made, and a network of levels is necessary, an additional control of the accuracy of the work is afforded by the polygonal closing of the level lines forming the net; that is, from the condition that on passing round the polygon and

arriving at the starting-point we should have the same height as at first. This assumes that the points of the net are in the same level surface and that error of closure depends upon errors of observation only. In the usual case, where the net is small and the country comparatively level, such an assumption is quite allowable. On the other hand, if the net is very large or the country mountainous, systematic sources of error, arising principally from the spheroidal form of the earth and the deviation of the plumb-line, would be introduced. The corrections resulting from these causes would have to be computed and applied. A full investigation of this point will be found in *Astron. Nachr.*, Vols. 80–84.

The adjustment of a net of levels may be carried out in a similar way to the adjustment of a triangulation. Thus suppose that the lines of levels form a closed figure. The conditions to be satisfied among the observed differences of height may be divided into two classes:

(a) Those arising from non-agreement of repeated measurements of differences of height between successive bench marks. The equations expressing these conditions correspond to the local equations in a triangulation.

(b) Those arising from the consideration that on starting from any bench mark and returning to it through a series of bench marks, thus forming a closed figure, we should find the original height. The resulting equations correspond to the angle and side equations in a triangulation.

If the circuit, instead of being a simple one, has tie lines, the number of closure conditions is easily estimated. For if s be the number of bench marks, and l the number of lines levelled over, the number of lines necessary to fix the bench marks is $s - 1$, and therefore the number of superfluous lines—that is, the number of closure conditions to be satisfied among the differences of height—is

$$l - s + 1.$$

174. *Approximate Methods of Adjustment.*—The differences of height between successive bench marks may be found

by taking the means of the observed values, as explained in Art. 172. Taking these means as independently observed quantities, they may be adjusted for non-satisfaction of closure conditions in various ways:

(a) The differences of level between the successive bench marks will give rise to l observation equations. Taking the starting point as origin, $s - 1$ differences are required to fix the s stations. If these $s - 1$ differences are considered independent unknowns the remaining $l - s + 1$ unknowns may be expressed in terms of them, and the solution completed by the method of Art. 109.

(b) Since each independent polygon in the net contains one superfluous measurement, it gives a condition equation. With s stations connected by l lines there must be $l - s + 1$ condition equations, as the number of superfluous measurements is $l - s + 1$. The solution may be completed by the method of correlates, Art. 110.

As to the relative advantages of the two forms of solution, if the adjusted values only are to be found, without their weights, the test will be furnished by the number of normal equations in each case, as the principal part of the labor consists in solving the normal equations. The number by the first form is $s - 1$, and by the second $l - s + 1$; and therefore, in an extensive net with few tie lines, the method of condition equations is to be preferred, and *vice versa*.

(c) Adjust each simple circuit in order by the principle of Art. 115, and repeat the process until the required accuracy is reached.

(d) The method employed on the British Ordnance Survey in reducing the principal triangulation. (See Art. 152.) This method is easily applied and gives results practically close enough with the first approximation. It deserves to be employed much more than it is in work of this kind.

The Ordnance Survey levels, however, were not reduced by this method. "The discrepancies in the levelling along the different lines or routes brought out in closing on

common points have been treated for England and Wales as a whole and rigorously worked out by the method of least squares, involving ultimately the solution of a system of equations with 91 unknown quantities."*

Ex. In the figure $AWMG$, A is the initial point, height zero, and the measured heights and distances are as follows:

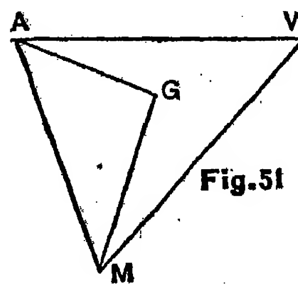


Fig. 51

$AW = \overset{m}{42.65101}$	$D_1 = 37.8$ kil.
$AM = 54.74663$	$D_2 = 35.8$ "
$AG = 58.56223$	$D_3 = 22.6$ "
$WM = 12.10530$	$D_4 = 44.2$ "
$MG = 3.79892$	$D_5 = 27.9$ "

required the adjusted values of the heights.

First Solution.—Let the most probable corrections to the five measured differences of height in order be v_1, v_2, v_3, v_4, v_5. The points W, M, G are completely determined by AW, AM, AG. We may, therefore, take v_1, v_2, v_3 as independent unknowns.

The observation equations are

$$\begin{aligned}
v_1 &= v_1 && \text{weight } 26.5 \\
v_2 &= v_2 && \text{"} \quad 27.9 \\
v_3 &= v_3 && \text{"} \quad 44.2 \\
v_4 &= -v_1 + v_2 - 0.00968 && \text{"} \quad 22.6 \\
v_5 &= -v_2 + v_3 + 0.01668 && \text{"} \quad 35.8
\end{aligned}$$

the weights being computed from $\dfrac{1000}{D}$.

The solution is finished as in first solution, Art. 140.

Second Solution.—From the closed circuits AWM, AMG we have the condition equations

$$\begin{aligned}
v_1 - v_2 + v_4 &= -0.00968 \\
v_2 - v_3 + v_5 &= +0.01668
\end{aligned}$$

with

$$\left[\frac{v^2}{D}\right] = \text{a min.}$$

The solution is finished as in second solution, Art. 140.

* *Abstract of Levelling in England and Wales.* Introduction, p. vi.

Third Solution.—Adjust each simple closed circuit in the figure in order. Since in the circuit AMG the algebraic sum of the three differences of height should be zero, we may apply the principles of Arts. 110, 111. We have

$$\begin{array}{rlll} AM = & 54.74663 & D_2 = 35.8 & \therefore\ p_2 = 27.9 \\ MG = & 3.79892 & D_5 = 27.9 & p_5 = 35.8 \\ AG = & -58.56223 & D_3 = 22.6 & p_3 = 44.2 \\ \hline & -\ 0.01668 & 86.3 & \end{array}$$

$$\text{Correction to } AM = \frac{35.8}{86.3} \times 0.01668 = +0.00692$$

$$\text{Weight of adjusted } AM = 27.9 + \frac{1}{\dfrac{1}{35.8} + \dfrac{1}{44.2}}$$
$$= 47.7$$

Hence for the circuit AMW

$$\begin{array}{ll} AM = 54.75355 & p_2' = 47.7 \\ AW = 42.65101 & p_1 = 26.5 \\ WM = 12.10530 & p_4 = 22.6 \end{array}$$

which may be adjusted as above.

The circuit AMG may be again adjusted, and so on.

Fourth Solution.

Weight.	A	W	M	G	Means.
			I.		
26.5	0.	42.65101			
27.9	0.		54.74663		
44.2	0.			58.56223	
22.6		0.	12.10530		
35.8			0.	3.79892	
			II.		
	0.	42.65101			
	0.		54.74663		
	0.			58.56223	
		42.65101	54.75631		
			54.75096	58.54988	
Means	0.	42.65101	54.75096	58.55670	
			III.		
	0.	0.			0.
	0.		+ 0.00433		+ 0.00216
	0.			− 0.00553	− 0.00276
		0.	− 0.00535		− 0.00268
			0.	+ 0.00682	+ 0.00341
			IV.		
	0.	42.65101			
	+ 0.00216		54.74879		
	− 0.00276			58.55947	
		42.64833	54.75363		
			54.75437	58.55329	
Means	− 0.00063	42.64978	54.75237	58.55670	
Final values	0.	42.65041	54.75300	58.55733	

Trigonometrical Levelling.

175. In extensive surveys in which a primary triangulation is carried on, the heights of the stations occupied may be conveniently found by trigonometrical levelling. The extra labor required to measure the necessary vertical angles while the horizontal angles are being read at the several stations is but slight.

When a country is hilly, heights can be perhaps as closely determined by trigonometrical levelling as by spirit levelling; but if the country is flat and the triangulation stations low, experience has shown that much more dependence is to be placed on the latter method. See, for comparisons of the relative accuracy of the two methods, Report of *U. S. Coast Survey*, 1876, App. 16 and 17; Report of *G. T. Survey of India*, vol. ii.; Report of *New York State Survey*, 1882. Sometimes, indeed, the agreement is so close that it must be regarded as accidental. Thus in the determination of the axis of the St. Gothard Tunnel Engineer Koppe found the difference of height of the two ends by trigonometrical levelling to be $39^m.13$. The spirit levelling of precision executed by Hirsch and Plantamour gave it $39^m.05$.

The great cause of inaccurate work in trigonometrical levelling is atmospheric refraction. This is one of those disturbing causes which is so erratic in its character that no method has yet been devised for determining it that is very satisfactory. Hence the plan adopted in trigonometrical levelling is to observe only at or near the time of minimum refraction. Without this precaution very discrepant results may be looked for. For example, in India, where in certain districts the triangulation has been carried for hundreds of miles over a level country with stations 10 or 12 miles apart and from 18 to 24 feet high—just high enough to be mutually visible at the time of minimum refraction—"numerous instances are recorded of the vertical angles varying through a range of 6 to 9 minutes, corre-

sponding to an apparent change in altitude of 100 to 150 feet in the course of 24 hours." *

176. *To Find the Zenith Distance of a Signal.*—Point the telescope at a mark on the signal and bisect it with the horizontal thread. Then turn the telescope 180° on its axis, transit it, and bisect the mark again. At both bisections read the vertical circle and the level parallel to the vernier arm. One-half of the difference of the circle readings corrected for level will give a single determination of the value of the zenith angle sought. The result is free from index error of the circle.

Errors of graduation with good instruments are so small in comparison with the uncertainties arising from refraction that it is unnecessary to eliminate them in the measure of vertical angles.

177. Let us consider the general case in which the zenith angles measured at two stations whose difference of height is required are not read simultaneously. In the figure, if A_1, A_2 denote the positions of the instruments employed on two lofty stations, the angles that the observer has attempted to measure are $Z_1A_1A_2$ and $Z_2A_2A_1$. But on account of the refractive power of the atmosphere the path of a ray of light from A_1 to A_2 will not be a straight line but a curve more or less irregular, and the direction in which A_2 is seen from A_1 will be that of the tangent A_1T to this curve at A_1. The line of sight from A_2 to A_1 will not necessarily be over the same curve.

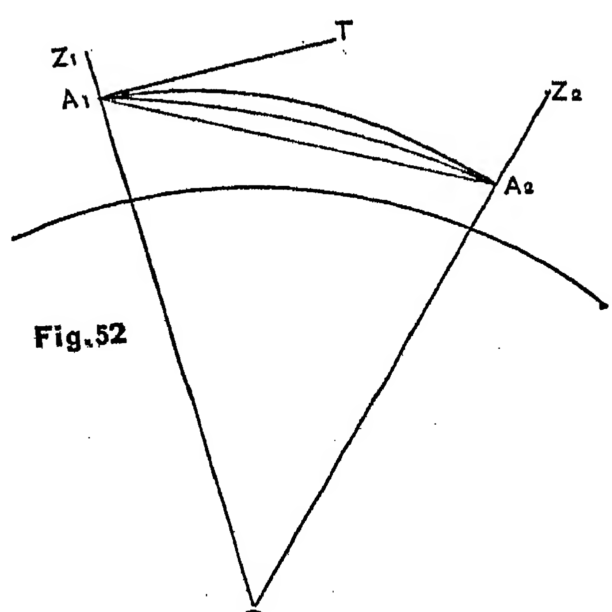


Fig. 52

The angles between the apparent directions of the rays of light at A_1, A_2 and the real directions are called *refraction angles*. Thus TA_1A_2 is the refraction angle at A_1.

* *Mem. Roy. Astron. Soc.*, vol. xxxiii. p. 104.

Let z_1, z_2 denote the measured zenith angles reduced to the heights of the instruments at A_1, A_2.

$D =$ the distance between the instruments.
$C =$ the angle at the earth's centre subtended by D.
$\zeta_1, \zeta_2 =$ the refraction angles at A_1, A_2.

Now, assuming the paths of the rays of light from one station to the other to be arcs of circles, which is approximately the case when the lines are of moderate length, we have

$$\zeta = \tfrac{1}{2}\frac{\text{path of ray}}{R_c}$$

where R_c is the radius of the refraction curve.

But approximately

$$C = \frac{\text{path of ray}}{R}$$

where R is the mean radius of the earth.

Hence we may put

$$\zeta_1 = \tfrac{1}{2}k_1 C \qquad \zeta_2 = \tfrac{1}{2}k_2 C \tag{1}$$

where k_1, k_2 are constants and may be called *refraction factors*.

Observations are frequently made so as to be simultaneous at A_1, A_2, and the line of sight may then be assumed to be the same arc of a circle for both directions. In this case if we put

$$\zeta_1 + \zeta_2 = kC$$

k is called the *coefficient of refraction*.

178. **To Find the Refraction Factors.**—The triangle CA_1A_2 gives the relation

$$180° - (z_1 + \zeta_1) + 180° - (z_2 + \zeta_2) + C = 180° \tag{1}$$

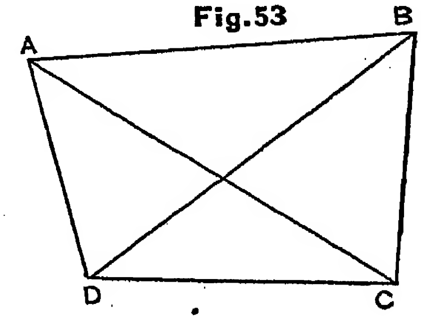
But, with sufficient accuracy, $C = \dfrac{D}{R \sin 1''}$, where R is the radius of curvature of the line in question.

Hence, attending to equations 1, Art. 177, we may write the above relation in the form

$$k_1 + k_2 = 2\left\{1 - \frac{R \sin 1''}{D}(z_1 + z_2 - 180°)\right\} \qquad (2)$$

This equation shows that a single line will not give the refraction factors, and we must, therefore, have a net of lines with the zenith angles read at the ends of each line. If, for simplicity, we consider a quadrilateral *ABCD* we shall have six equations of the form (2). As these equations contain twelve unknowns we may, in order to reduce this number, assume, if the observations at station *A* over the lines *AB*, *AC*, *AD* are nearly simultaneous, that the *k* for each of these lines—that is, the *k* for station *A* —is the same. Similarly at stations *B*, *C*, *D*, so that in the most favorable case we shall have 4 unknowns and 6 equations.

Fig.53



It may not always be possible to get one *k* for each station, but in a net, if all of the lines have been sighted over, a sufficient excess in number of equations over unknowns will in general be found to admit of solution by the method of least squares. As regards the weights to be assigned to these equations, we may proceed in the ordinary way. If z_1 were observed n_1 times, and z_2 were observed n_2 times, the weights of z_1 and z_2 may be taken to be n_1 and n_2 respectively, and the first equation would have a weight P given by

$$\frac{1}{P} = \frac{4R^2 \sin^2 1''}{D^2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

that is, its weight would be proportional to

$$\frac{n_1 n_2}{n_1 + n_2} D^2$$

179. **To Find the Mean Coefficient of Refraction.**—When the zenith distances are simultaneous,

$$k_1 = k_2 = k \text{ suppose}$$

and the refraction coefficient at the moment of observation is for both stations,

$$k = 1 - \frac{R \sin 1''}{D}(z_1 + z_2 - 180°)$$

A value of k can thus be found for each line sighted over from both ends simultaneously. To get an average value of k for the whole system the weighted mean of these separate values must be taken. The same method of assigning weights may be used as in the preceding. Bessel argues that errors arising from irregularities in k are of much more importance than the errors in the zenith angles, and proposes the empirical formula*

$$\frac{n_1 n_2}{n_1 + n_2} \sqrt{D}$$

for assigning relative weights. In this he is followed by the Coast Survey in their determination of the coefficient of refraction from observations made in Northern Georgia near Atlanta base.

The mean values of k found by the Coast Survey in N. Georgia (1873) and by the New York State Survey (1882) were 0.143 and 0.146 respectively.

180. **To Find the Differences of Height.**—There are three cases to be considered:

(1) When the zenith angles at both ends of each line are used.

(a) When the zenith angles are not simultaneous.

Let H_1 = known height of first station
H_2 = height of next station

* *Gradmessung in Ostpreussen*, p. 196.

From the triangle CA_1A_2 (Fig. 52)

$$\frac{H_2 - H_1}{H_2 + H_1 + 2R} = \frac{CA_2 - CA_1}{CA_2 + CA_1} = \frac{\tan\frac{1}{2}(A_1 - A_2)}{\tan\frac{1}{2}(A_1 + A_2)} \qquad (1)$$

Substituting for A_1, A_2 their values in terms of z_1, z_2, k_1, k_2, and reducing,

$$H_2 - H_1$$
$$= \left\{ D \tan \tfrac{1}{2}(z_2 - z_1) + \frac{k_2 - k_1}{4R} D^2 \right\} \left\{ 1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2} \right\} \quad (2)$$

(b) When the zenith angles are simultaneous $k_1 = k_2$, and

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1)\left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2}\right) \qquad (3)$$

which is the form used on the Coast Survey.

(2) When the angles observed at each end of a line are used separately.

In the common case of a line sighted over from one end only, we have, since

$$A_1 = 180^\circ - z_1 - \zeta_1$$
$$A_2 = \qquad z_1 + \zeta_1 - C$$

by substituting these values in (1) and reducing,

$$H_2 - H_1$$
$$= D \cot\left\{ z_1 - (1 - k)\frac{D}{2R \sin 1''} \right\}\left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2}\right)$$

$$= D \cot z_1 + \frac{1 - k}{2R} D^2 + \frac{2 - k}{2R} D^2 \cot^2 z_1 \qquad (4)$$

which is equivalent to the Coast Survey form.

When the line is sighted over from both ends we have two values of the difference of height, whose weighted mean gives the required result.

181. **Precision of the Formulas for Differences of Height.**—If h_1, h_2, h_3 denote the differences of height between two stations found from non-simultaneous readings of angles at the stations, from simultaneous readings, and from angles read at one of the stations only, respectively, then, with sufficient accuracy,

$$
\begin{aligned}
h_1 &= D \tan \tfrac{1}{2}(z_2 - z_1) + (k_2 - k_1)\frac{D^2}{4R} \\
h_2 &= D \tan \tfrac{1}{2}(z_2 - z_1) \qquad\qquad (1) \\
h_3 &= D \cot z_1 + (1 - k_1)\frac{D^2}{2R}
\end{aligned}
$$

Let the m. s. e. of an observed zenith angle be μ_z. Denote by μ_{h_1}, μ_{h_2}, μ_{h_3} the m. s. e. of the differences of height found, and by μ_k the m. s. e. of a refraction factor k. Then by differentiation, taking D, z_1, z_2, k_1, k_2 as independent variables, and remembering that z_1, z_2 are each equal to 90° nearly, and that we may put $dD = 0$, since the distances are well known in comparison with the heights, we shall have (Art. 65)

$$
\begin{aligned}
\mu_{h_1}^2 &= \tfrac{1}{2} D^2 \sin^2 1'' \, \mu_z^2 + \tfrac{1}{8}\frac{D^4}{R^2}\mu_k^2 \\
\mu_{h_2}^2 &= \tfrac{1}{2} D^2 \sin^2 1'' \, \mu_z^2 \qquad\qquad (2) \\
\mu_{h_3}^2 &= D^2 \sin^2 1'' \, \mu_z^2 + \tfrac{1}{4}\frac{D^4}{R^2}\mu_k^2
\end{aligned}
$$

These results show that differences of height are found with the greatest precision from simultaneous observations.

182. **Adjustment of a Net of Trigonometric Levels.**—It will be sufficient to consider a simple closed figure, as a net of levels can be broken up into a number of closed figures, generally triangles. For simplicity take a triangle.

The same value of k is assumed for all of the lines radiating from a station. Denote the values of k at the three stations by k_1, k_2, k_3 respectively. Then we have the observation equations

$$k_1 + k_2 = 2\left\{1 - \frac{R\sin 1''}{D_1}(z_1 + z_2 - 180°)\right\} = l_1, \text{ a known quan.}$$

$$k_2 + k_3 = 2\left\{1 - \frac{R\sin 1''}{D_2}(z_2 + z_3 - 180°)\right\} = l_2, \quad \text{"} \quad \text{"}$$

$$k_3 + k_1 = 2\left\{1 - \frac{R\sin 1''}{D_3}(z_3 + z_1 - 180°)\right\} = l_3, \quad \text{"} \quad \text{"}$$

The condition to be satisfied among the differences of height at the vertices of the triangle is that, on starting from any station and returning to it through the other two stations, we should find the original height. Thus proceeding round the triangle in order of azimuth, if h_1, h_2, h_3 denote the differences of height of the stations, and (h_1), (h_2), (h_3) the most probable corrections to these values, we must have

$$h_1 + (h_1) + h_2 + (h_2) + h_3 + (h_3) = 0$$

that is, we must have a condition equation of the form

$$ak_1 + bk_2 + ck_3 = l_4$$

where a, b, c, l_4 are constants.

The four equations may be solved by the method of correlates, and the differences of height may next be computed from equations 1, Art. 181, and will be found consistent.

If the circuit, instead of being a simple one, has diagonals, then, as in Art. 173, if l is the number of lines read over, and s the number of stations in the circuit, the number of conditions to be satisfied among the differences of height is

$$l - s + 1$$

Ex. In the triangulation of Georgia, near Atlanta Base, there were measured zenith angles as follows:*

Stations occupied.	Station read to.	Zenith Angle.	Time.
S. W. Base,	Sweat Mt.	89° 37′ 53″.7	2 days.
	Stone Mt.	89° 26′ 21″.6	
N. E. Base,	Sweat Mt.	89° 39′ 51″.7	1 "
	Stone Mt.	89° 24′ 04″.0	
Stone Mt.,	N. E. Base.	90° 43′ 20″.8	2 "
	S. W. Base.	90° 41′ 56″.8	
	Sweat Mt.	90° 08′ 58″.9	
Sweat Mt.,	S. W. Base.	90° 33′ 37″.2	1 "
	N. E. Base.	90° 31′ 43″.4	
	Stone Mt.	90° 09′ 37″.0	

The mean latitude is 34° N. approx.

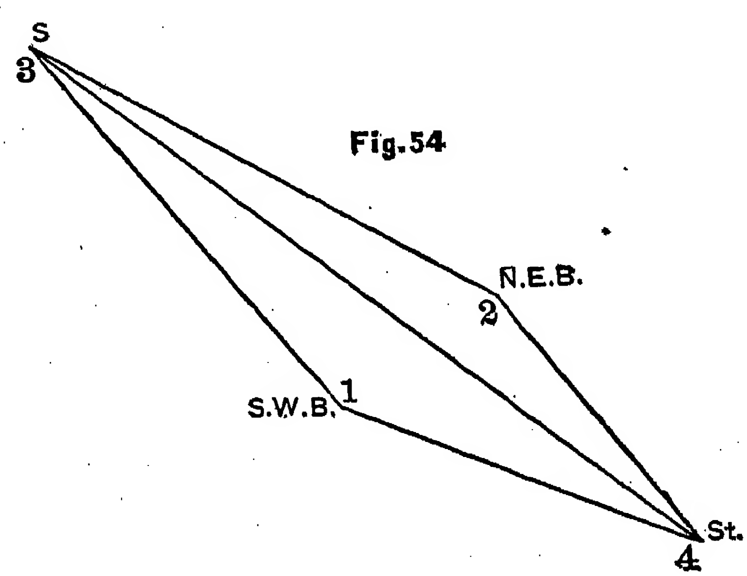


Fig. 54

Distances.

	m.
S. W. Base–Sweat Mt.,	24368
N. E. Base–Sweat Mt.,	25203
N. E. Base–Stone Mt.,	16321
Stone Mt.–S. W. Base,	17761
Sweat Mt.–Stone Mt.,	40726

Call k_1, k_2, k_3, k_4 the refraction factors for all lines at the stations 1, 2, 3, 4 respectively.

* The measures are taken from *C. S. Report*, 1876.

We must form two classes of equations:

(1) Observation equations. These are computed from the form

$$k + k' = 2\left\{1 - \frac{1}{BD}(z + z' - 180°)\right\}$$

where $B = \dfrac{1}{R \sin 1''}$

The weights are computed according to the formula $\frac{n_1 n_2}{n_1 + n_2} \sqrt{D}$ (Art. 179). It will be close enough to take n_1, n_2 as the number of days of observation. That is the best we can do with our data. We have then the observation equations

$$\begin{array}{lll} k_1 + k_3 = 0.249 & \text{weight} & 1.04 \\ k_1 + k_4 = 0.267 & \text{"} & 1.33 \\ k_3 + k_4 = 0.308 & \text{"} & 1.35 \\ k_2 + k_3 = 0.297 & \text{"} & 0.80 \\ k_2 + k_4 = 0.317 & \text{"} & 0.85 \end{array}$$

(2) Condition equations.

$$\begin{array}{rl} \text{The number of lines} & = 5 \\ \text{The number of stations} & = 4 \\ \therefore \text{ The number of condition equations} & = 5 - 4 + 1 \\ & = 2 \end{array}$$

These two condition equations arise from the sums of two sets of three equations, each of the form

$$H - H' = D \tan \tfrac{1}{2}(z - z') + \frac{D^2}{4R}(k - k')$$

They are, for the triangles 134, 234,

$$\begin{aligned} 0 &= -1.52 - 10.92\,k_1 - 41.79\,k_3 + 52.71\,k_4 \\ 0 &= +1.84 + 14.47\,k_2 + 40.17\,k_3 - 54.64\,k_4 \end{aligned}$$

The solution of these equations, subject to the relation

$$[kk] = \text{a minimum},$$

gives

$$\begin{array}{ll} k_1 = 0.118 & k_3 = 0.149 \\ k_2 = 0.107 & k_4 = 0.171 \end{array}$$

whence the adjusted differences of height follow at once:

$$\begin{array}{lr} & m \\ 1 \text{ to } 3 & +198.24 \\ 3 \text{ to } 4 & -\quad 2.30 \\ 4 \text{ to } 1 & -195.94 \end{array}\Bigg\} \text{ check sum } \quad 0.00$$

$$\begin{array}{lr} 3 \text{ to } 2 & -191.17 \\ 2 \text{ to } 4 & +188.88 \\ 4 \text{ to } 3 & +\quad 2.30 \end{array}\Bigg\} \text{ check sum } +0.01$$

The precision of the adjusted values, or of any function of them, may be found exactly as in Art. 114.

183. **Approximate Method of Adjusting a Net.**—On account of the many uncertainties attendant on finding the refraction factors, it is not often that so elaborate a method of adjusting the heights as the preceding is followed.

In the ordinary method of observation, where the observed zenith angles are simultaneous at every two stations,

$$h = H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1)$$

and the differences of height may be computed at once without any reference to the coefficient of refraction. These differences of height, considered as observed quantities, may be adjusted for conditions of closure in the net, as in spirit levelling, Art. 174.

The weights P to be assigned to the differences of height in the solution will be found from

$$\frac{1}{P} = \mu_h^2 = \tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2$$

and therefore the weights are inversely proportional to the squares of the distances between the stations.

When the zenith angles are not simultaneous, after finding the refraction factors, as in Art. 182, and computing the differences of height, we should find the weights of these differences of height from

$$\frac{1}{P} = \tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2 + \tfrac{1}{8} \frac{D^4}{R^2} \mu_k^2$$

It would seem safe to assume $\mu_z = 2''$, $\mu_k = 0.02$.

Now,
$$\tfrac{1}{2} D^2 \sin^2 1'' \mu_z^2 \gtrless \tfrac{1}{8} \frac{D^4}{R^2} \mu_k^2$$

as
$$D \lesseqgtr 4 \text{ miles.}$$

Hence for distances between the stations up to 4 miles the first term is the important one, and for greater distances

the second term. We should, therefore, for distances between stations not greater than 4 miles, weight inversely as the square of the distance, and for distances over that amount inversely as the fourth powers of the distances.

Ex. 1. In the determination of the axis of the St. Gothard Tunnel (Fig. 37) the heights of the trigonometrical stations were determined by trigonometrical levelling. The following were the results unadjusted with their weights. Required the values adjusted for closure of circuits.

	Diff. of height.	Wt.		Diff. of height.	Wt.
	m			m	
Airolo– XII.	914.96	23	III.– IX.	216.46	1
Airolo– X.	1287.75	17	V.– IX.	899.87	1
Airolo– XI.	1299.27	2	V.–VIII.	1075.77	1
Airolo– IX.	1553.09	5	III.–VIII.	391.74	1
XII.– X.	372.73	5	VII.–VIII.	901.78	1
XII.– XI.	384.41	2	V.– VII.	174.45	7
XII.– IX.	638.30	3	IV.– III.	296.69	60
XII.–VIII.	814.35	1	VII.– III.	509.49	4
X.– XI.	11.60	3	Göschenen– III.	1376.19	14
X.– IX.	265.48	6	V.– IV.	387.24	20
X.–VIII.	441.10	2	VII.– IV.	212.75	7
XI.– IX.	253.87	1	Göschenen– IV.	1079.50	30
XI.–VIII.	429.55	10	Göschenen– V.	692.35	15
IX.–VIII.	175.37	1			

The heights of Airolo and Göschenen are $1147^{m}.12$ and $1108^{m}.07$ respectively, as found by spirit levelling, and are unaltered in the adjustment. The adjusted heights of the stations will be found to be

	m		m
III.	2484.26	IX.	2700.35
IV.	2187.57	X.	2434.87
V.	1800.36	XI.	2446.49
VII.	1974.78	XII.	2062.08
VIII.	2876.07		

Ex. 2. From

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1)\left(1 + \frac{H_2 + H_1}{2R} + \frac{D^2}{12R^2}\right)$$

deduce

$$H_2 - H_1 = D \tan \tfrac{1}{2}(z_2 - z_1) + \frac{H_2^2 - H_1^2}{2R} + \frac{D^2}{12R^2}(H_2 - H_1)$$

Ex. 3. If a series of points connected by observed zenith angles begin and end with points whose heights are known, then the number of conditions to be satisfied among the differences of height is

$$l - s + 2$$

where l is the number of lines read over and s the number of points in the net.

Ex. 4. If zenith angles are read to two stations from a station between, show that the difference of height, h, of the two stations will be found from

$$h = D_2 \cot z_2 - D_1 \cot z_1 - \frac{D_2^2 - D_1^2}{2R} k$$

where D_1, D_2 are the distances from the station occupied.

Also show that when $D_1 = D_2$ the precision of the height h is the same as if the observations had been made simultaneously at the two extreme stations themselves.

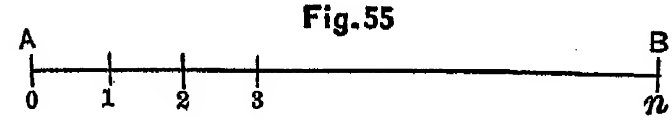
CHAPTER IX.

APPLICATION TO ERRORS OF GRADUATION OF LINE MEASURES AND TO CALIBRATION OF THERMOMETERS.

Observations for the determination of errors of graduation of line measures, and observations for the calibration of thermometers, may be discussed together, as there is no essential difference in the method of reducing them.

184. **Line Measures.**—Let AB be a line measure divided into n equal parts as nearly as may be at the points $1, 2, \ldots n-1$, and let 0 and n be the initial and terminal marks. Comparisons are supposed to have been made between AB and a standard of length, so that the distance between 0 and n is known.

Fig.55



The problem proposed is to find the corrections to the intermediate graduation marks—that is, the amounts by which the positions of the marks should be changed to be in their true relative positions. When the corrections have been applied to the distances $0\,1$, $1\,2, \ldots$ these distances should be all the same proportional part of the entire distance AB.

In making the necessary observations two methods are in common use.

(a) Two microscopes furnished with micrometers are firmly mounted on a frame separate from the support on which the graduated line measure rests. The zeros of the micrometers are placed at a distance apart as nearly as possible equal to the distances to be read. The marks $0, 1$; $1, 2$; $\ldots$ are in succession brought under the microscopes and the micrometers read in each position. Each distance is thus compared with the constant distance be-

tween the micrometer zeros, which differs from the distance of the true positions of the marks by a fixed but unknown amount.

Let $x_0, x_1, \ldots x_n$ denote the corrections to the graduation marks at 0, 1, . . . n. Then for the first interval 0 1, if M_0, M_1 be the readings at 0, 1 and c be the unknown constant distance between the micrometer zeros, and d the distance between the corrected positions of the graduation marks, we should have

$$c + (M_1 + x_1) - (M_0 + x_0) = d$$

This may be written

$$x_0 - x_1 - y = l_{0\,1} \tag{1}$$

where

$$l_{0\,1} = M_1 - M_0, \text{ and } y = c - d$$

Hence, taking four spaces only,

$$\begin{array}{rrrrr} x_0 - x_1 & & & -y = & l_{0\,1} \\ & x_1 - x_2 & & -y = & l_{1\,2} \\ & & x_2 - x_3 & -y = & l_{2\,3} \\ & & & x_3 - x_4 - y = & l_{3\,4} \end{array} \tag{2}$$

But as the distance between the initial and terminal points is known, the corrections x_0 and x_4 may each be taken equal to zero. Also, since the equations contain four unknowns and are themselves only four in number, we must either reduce the number of unknowns arbitrarily or make additional observations involving other combinations of the unknowns, in order to apply the method of least squares.

It is usually more convenient to solve these equations by computing the corrections to the intervals 0 1, 0 2, . . . first of all, and then the corrections to the positions of the marks. Writing z_1 for $x_1 - x_0$, z_2 for $x_2 - x_1$, . . ., and eliminating y,

we have from the above equations, supposing the first space compared with all of the others,

$$\begin{aligned} z_1 - z_2 &= l_{12} - l_{01} \\ z_1 - z_3 &= l_{23} - l_{01} \\ z_1 - z_4 &= l_{34} - l_{01} \end{aligned} \qquad (3)$$

But

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 &= x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + x_4 - x_3 \\ &= 0 \end{aligned} \qquad (4)$$

Hence

$$\begin{aligned} z_1 &= \tfrac{1}{4}(l_{12} + l_{23} + l_{34} - 3l_{01}) \\ z_2 &= \tfrac{1}{4}(l_{01} + l_{23} + l_{34} - 3l_{12}) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \qquad (5)$$

Also $x_1 = z_1$, $x_2 = z_1 + z_2$, . . ., and are therefore known.

Ex. In order to find the corrections to the double decimetre graduation marks needed to change the nominal into exact values in terms of the interval $0 - 1^m$ on a Repsold steel metre comparisons were made as follows: Two microscopes were mounted at a distance of $0^m.2$ approx. from each other, and readings made by pointing at the successive double decimetre marks, the microscopes remaining stationary while the metre was run under them. The order of reading was $0^m.0$ and $0^m.2$; $0^m.2$ and $0^m.4$; . . . $0^m.8$ and $1^m.0$; $0^m.0$ and $0^m.2$, the interval $0^m.0$ and $0^m.2$ being read on at the beginning and end of each set of comparisons. Twenty-four sets were made and the following results obtained.*

	m	m	m	m	μ	μ
Interval,	0.2 to 0.4	= 0.0 to 0.2	+ 2.1	± 0.1
"	0.4 to 0.6	= 0.0 to 0.2	+ 2.6	± 0.1
"	0.6 to 0.8	= 0.0 to 0.2	+ 0.7	± 0.1
"	0.8 to 1.0	= 0.0 to 0.2	+ 2.2	± 0.1

With the above notation

$$\begin{aligned} z_1 - z_2 &= \overset{\mu}{2.1} \\ z_1 - z_3 &= 2.6 \\ z_1 - z_4 &= 0.7 \\ z_1 - z_5 &= 2.2 \end{aligned}$$

$$\begin{aligned} \therefore\ z_1 &= +\overset{\mu}{1.5}, \text{ and } & x_1 &= \overset{\mu}{1.5} \\ z_2 &= -0.6 & x_2 &= 0.9 \\ &\cdot\ \cdot\ \cdot\ \cdot & &\cdot\ \cdot\ \cdot \end{aligned}$$

* The value of the interval $0 - 1^m$ was found, by comparing with the German standard, to be $1^m + 86^\mu.18 + 10^\mu.65\, t$, where t is the temperature in degrees centigrade.

or in tabular form,

Graduation	$0^m.0$	0.2	0.4	0.6	0.8	$1^m.0$
Correction	$0^\mu.0$	-1.5	-0.9	$+0.1$	-0.7	$0^\mu.0$

The Precision.—Each of the intervals is entangled with the interval 0.0 and 0.2, and, therefore, the p. e. given are not independent. From the mode of measurement it is evident that the $(\text{p. e.})^2$ of 0.0 to 0.2 is half that of the $(\text{p. e.})^2$ of each of the other intervals. Hence if r_1 is the p. e. of the first interval, and r the p. e. of each of the others,

$$r_1^2 = 2r^2$$

$$r_1^2 + r^2 = (0.1)^2$$

$$\therefore\ r^2 = 0.007 \qquad\qquad r_1^2 = 0.003$$

Hence

$$r_{x1}^2 = \tfrac{1}{25}(4 \times 0.007 + 16 \times 0.003)$$

and

$$r_{x1} = \pm 0^\mu.06$$

Similarly for the other marks.

From equations 2 definite values of the corrections may be found if x_0 and x_4 are supposed known. There is, however, a greater probability of eliminating systematic error if, instead of spending all the time of observation in the direct comparison of the single intervals, we spend part of it in comparing combinations of those intervals. Let, then, as the best arrangement (see Art. 153), the single intervals 0 1, 1 2, 2 3, 3 4, and all possible combinations of intervals, as 0 2, 1 3, 2 4; 0 3, 1 4, be equally well compared. The microscope intervals would, of course, be different in each set of comparisons, being approximately 0 1, 0 2, 0 3. The observation equations for this arrangement are

$$\begin{array}{lllllllllll}
x_0 & -x_1 & & & & -y_1 & & & & = l_{01} & \\
& x_1 & -x_2 & & & -y_1 & & & & = l_{12} & \\
& & x_2 & -x_3 & & -y_1 & & & & = l_{23} & \\
& & & x_3 & -x_4 & -y_1 & & & & = l_{34} & \\
x_0 & & -x_2 & & & & -y_2 & & & = l_{02} & \quad (6) \\
& x_1 & & -x_3 & & & -y_2 & & & = l_{13} & \\
& & x_2 & & -x_4 & & -y_2 & & & = l_{24} & \\
x_0 & & & -x_3 & & & & -y_3 & & = l_{03} & \\
& x_1 & & & -x_4 & & & -y_3 & & = l_{14} & \\
x_0 & & & & -x_4 & & & & -y_4 & = l_{04} &
\end{array}$$

But x_0 and x_4 are known. Hence the normal equations

$$
\begin{array}{lr}
+4x_1 - x_2 - x_3 \qquad + y_2 + y_3 = -l_{01} + l_{12} + l_{13} + l_{14} & \\
- x_1 + 4x_2 - x_3 \qquad\qquad = -l_{02} - l_{12} + l_{23} + l_{24} & \\
- x_1 - x_2 + 4x_3 \qquad - y_2 - y_3 = -l_{03} - l_{13} - l_{23} + l_{34} & \\
\qquad + 4y_1 \qquad\qquad = -l_{01} - l_{12} - l_{23} - l_{34} & (7) \\
+ x_1 \qquad - x_3 \qquad + 3y_2 \qquad = -l_{02} - l_{13} - l_{24} & \\
+ x_1 \qquad - x_3 \qquad\qquad + 2y_3 = -l_{03} - l_{14} &
\end{array}
$$

from which equations the most probable values of the corrections may be found.

The Precision of the Corrections x_1, x_2, x_3.—The m. s. e. of an observation of the unit of weight is found from the usual formula. We have

$$\mu = \sqrt{\frac{[vv]}{9-6}}$$

the number of observations being 9, and of independent unknowns 6.

The weights of x_1, x_2, x_3 may be found by the methods of Chapter IV.

The weight of x_1 and of x_3 is $\frac{20}{7}$, and the weight of x_2 is $\frac{20}{6}$. Hence the m. s. e. of x_1, x_2, x_3 are known.

Ex. **Solve by this method the example in Art. 187.**

(b) The work of reduction is much facilitated by employing an auxiliary scale, CD, divided into spaces approximately equal to those of AB, and whose values have already been found by comparison with some standard. If, as before, we suppose the single spaces compared, and also all possible combinations of spaces, the observation equations would be the same as equations 6. The second scale, however, enables us to find each microscope interval employed. Hence y_1, y_2, . . . are known, and may, therefore, be transposed in these equations and added to the terms l_{01}, l_{12}, . . . The observation equations thus involve only

$x_0, \ldots x_4$ as unknowns. Taking $x_0 = 0$, $x_4 = 0$, the normal equations are

$$\begin{aligned}
+4x_1 - x_2 - x_3 &= -l_{01} + l_{12} + l_{13} + l_{14} \\
-x_1 + 4x_2 - x_3 &= -l_{02} - l_{12} + l_{23} + l_{24} \\
-x_1 - x_2 + 4x_3 &= -l_{03} - l_{13} - l_{23} + l_{34}
\end{aligned} \tag{8}$$

from which x_1, x_2, x_3 may be found.

The reduction may, however, be made more easily by the application of the artifice employed in Art. 156. If we consider x_0 and x_4 as yet unknown, the normal equations may be written

$$\begin{aligned}
4x_0 - x_1 - x_2 - x_3 - x_4' &= l_{01} + l_{02} + l_{03} + l_{04} \\
-x_0 + 4x_1 - x_2 - x_3 - x_4 &= -l_{01} + l_{12} + l_{13} + l_{14} \\
-x_0 - x_1 + 4x_2 - x_3 - x_4 &= -l_{02} - l_{12} + l_{23} + l_{24} \\
-x_0 - x_1 - x_2 + 4x_3 - x_4 &= -l_{03} - l_{13} - l_{23} + l_{34} \\
-x_0 - x_1 - x_2 - x_3 + 4x_4 &= -l_{04} - l_{14} - l_{24} - l_{34}
\end{aligned} \tag{9}$$

Adding these equations, there results

$$0 = 0$$

This was to be expected, because we have left the initial and terminal points free. Hence we may assume any arbitrary relation between the corrections. Let us assume as most convenient

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0 \tag{10}$$

and then by adding this relation to each of the normal equations we have

$$\begin{aligned}
5x_0 &= +l_{01} + l_{02} + l_{03} + l_{04} \\
5x_1 &= -l_{01} + l_{12} + l_{13} + l_{14} \\
5x_2 &= -l_{02} - l_{12} + l_{23} + l_{24} \\
5x_3 &= -l_{03} - l_{13} - l_{23} + l_{34} \\
5x_4 &= -l_{04} - l_{14} - l_{24} - l_{34}
\end{aligned} \tag{11}$$

The whole solution may be conveniently arranged in tabular form. The sums of the horizontal rows are first found and then placed in the proper vertical columns, with

signs changed. The vertical columns are next added and divided by 5.

	$-l_{01}$	$-l_{02}$	$-l_{03}$	$-l_{04}$	Sum_1
		$-l_{12}$	$-l_{13}$	$-l_{14}$	Sum_2
			$-l_{23}$	$-l_{24}$	Sum_3
				$-l_{34}$	Sum_4
$-\mathrm{Sum}_1$	$-\mathrm{Sum}_2$	$-\mathrm{Sum}_3$	$-\mathrm{Sum}_4$		
$5x_0$	$5x_1$	$5x_2$	$5x_3$	$5x_4$	
x_0	x_1	x_2	x_3	x_4	

By putting $x_0 = 0$, $x_4 = 0$ in equations 11, it is easily seen that we have the same results for x_1, x_2, x_3 as found from equations 8.

Ex. The intervals 0–5 mm., 5–10 mm., . . . 25–30 mm. on a steel metre were compared with the interval 92–94 hundredths on a standard inch. The whole intervals 0–30 mm. being known from other comparisons, show that the p. e. of the interval 0–5 mm., if the p. e. of a comparison is r, is $\frac{r}{6}\sqrt{30}$.

[Call metre intervals s_1, s_2, . . . s_6, and inch interval a. Then

$$s_1 = a + \Delta_1$$
$$s_2 = a + \Delta_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$s_6 = a + \Delta_6$$

where Δ_1, Δ_2, . . . Δ_6 are the errors.

By addition,

$$[s] = 6a + [\Delta]$$

$$\therefore \quad a = \frac{[s]}{6} - \frac{[\Delta]}{6}$$

where $[s]$ is a known quantity.

Hence eliminate a from the value of s_1.]

185. **Calibration of Thermometers.**—In Fig. 55 let 0, 1, 2, . . . n denote the graduation marks cut on the glass stem, AB, of a thermometer. The point A we will take to be the freezing point and B the boiling point. These points are known, being first determined by the maker of the in-

strument, and can be redetermined at any time by special experiments.

We will suppose that the errors of the freezing and boiling point marks are known, and proceed to consider the corrections to the intermediate marks due to want of uniformity of the bore of the stem, or the *calibration* corrections, as they are called. If the bore between two assigned marks were uniform it could be filled by a certain volume of mercury. The length of this standard volume or column would be indicated by the readings at the two marks. If the column were moved along until it came between two other marks with the same difference of readings as before, and the bore were not uniform with the bore at first position, the column would not fill the stem between the marks. The amount of difference would be the calibration correction for this interval.

In explaining the method of determining the calibration corrections let us for simplicity consider 3 intermediate points only between the freezing and boiling points; that is, 4 intervals, 0 1, 1 2, 2 3, 3 4, of 45° each on Fahrenheit's scale. A column of mercury, of volume sufficient to fill 0 1 as nearly as may be, is broken off and the ends read when in the positions 0 1, 1 2, 2 3, 3 4. Another, equal to 0 2, is read in the positions 0 2, 1 3, 2 4, and a third, equal to 0 3, in the positions 0 3, 1 4.*

As it is impossible to break off the exact column required in every case, we break off one as nearly equal to it as possible and neglect the error introduced by the small discrepancy, which need not exceed 0°.2.

186. The following method of breaking off column lengths was that employed by Mr. Charles C. Brown, of the Lake Survey, in determining the calibration corrections of thermometer 5280 Green (New York):

A column of mercury 200° to 250° in length was easily obtained by making that amount of mercury run into the

* This method of making the readings is known as Neumann's. See the corresponding form in triangulation Art. 154.

stem, a few slight jars being sufficient to start the column moving; a little manipulation then brought the empty space in the bulb to the junction of the bulb and the stem, when a sudden turn of the thermometer upright broke off the column, and almost as sudden an inversion preserved it. When too large a space was left in the bulb to be filled by heating the thermometer to 140° or 150°, a few drops of mercury were allowed to drop off the end of the column in the stem held upright, good care being taken to stop the operation before the column joined the mercury in the bulb. Then, the thermometer being heated until the mercury from the bulb began to appear in the stem, the column already in the stem was run down carefully, and partially joined to the mercury in the bulb, leaving a small bubble on one side of the column, the thermometer being allowed to cool slowly until the desired length of column above this bubble (which remains very nearly stationary) was obtained. The column was broken off at the bubble by a slight twitch or jar. If there are objections to heating the thermometer above a certain temperature, column lengths above 10° to 20° or 30° longer than the number of degrees of that temperature, depending on the distance of the 32° point from the bulb, can be obtained by jarring off small drops of mercury from a long column into the reservoir at the top of the stem. It requires much more time, care, and patience to obtain a column in this way than in the other. Columns more than about 160° in length were so obtained in this case. It is rather difficult to break off short columns 5° to 15° in length in the manner first described, the weight of mercury in the short column not giving momentum enough to move it away from the rest of the column readily. A little patience is all that is necessary, however. To be able to read the shorter columns at 32° a column 10° to 50° long, depending on the temperature at the time, must be broken off and put into the reservoir at the upper end of the tube, out of the way.

187. Let x_0, x_1, x_2, x_3, x_4 denote the calibration correc-

tions at the several graduation marks. Then, approximately, we have for each interval a relation of the form

$$\text{column} = \text{diff. of readings} + \text{diff. of cal. corr.}$$

As the volumes of the columns broken off correspond to the constant interval between the microscopes in the case of line measures, the observation equations furnished by the thermometer readings will correspond in form to equations 6, Art. 184.

The rigorous solution of these equations is, however, rather complicated, and though in comparisons of standards of length it may be allowable, from the precision with which measurements can be made, to spend the labor demanded by this form of reduction, yet in thermometric work, where the readings cannot be very close from the nature of the case, an approximate form is sufficient. By the following artifice, which is quite analogous to that introduced by Hansen in the adjustment of a triangulation (Art. 155), the labor is reduced very materially.

Instead of finding the corrections to the graduation marks directly, the corrections to the several intervals between the graduation marks may first be found, and thence the corrections to the marks at the ends of the intervals. Thus if z_1, z_2, z_3, z_4 denote the corrections to the 4 intervals in our example, then

$$\begin{aligned} z_1 &= x_1 - x_0 \qquad & z_3 &= x_3 - x_2 \\ z_2 &= x_2 - x_1 \qquad & z_4 &= x_4 - x_3 \end{aligned} \qquad . \qquad (1)$$

From the observation equations 6 we have, by subtracting in pairs,

$$\begin{array}{rrrrl} z_1 & - z_2 & & & = l_{12} - l_{01} \\ & z_2 & - z_3 & & = l_{23} - l_{12} \\ & & z_3 & - z_4 & = l_{34} - l_{23} \\ z_1 & & - z_3 & & = l_{13} - l_{02} \\ & z_2 & & - z_4 & = l_{24} - l_{13} \\ z_1 & & & - z_4 & = l_{14} - l_{03} \end{array} \qquad (2)$$

Hence, considering $l_{12} - l_{01}$, $l_{23} - l_{12}$, . . . as independently observed quantities, we have the normal equations

$$\begin{aligned}
3z_1 - z_2 - z_3 - z_4 &= (l_{12} - l_{01}) + (l_{13} - l_{02}) + (l_{14} - l_{03}) \\
- z_1 + 3z_2 - z_3 - z_4 &= -(l_{12} - l_{01}) + (l_{23} - l_{12}) + (l_{24} - l_{13}) \\
- z_1 - z_2 + 3z_3 - z_4 &= -(l_{23} - l_{12}) + (l_{34} - l_{23}) - (l_{13} - l_{02}) \\
- z_1 - z_2 - z_3 + 3z_4 &= -(l_{34} - l_{23}) - (l_{24} - l_{13}) - (l_{14} - l_{03})
\end{aligned} \tag{3}$$

which equations when added give $0 = 0$ identically. The reason is that we have not yet fixed the initial or terminal points. To do this we may, as most convenient, assume the relation

$$z_1 + z_2 + z_3 + z_4 = 0 \tag{4}$$

Adding this relation to each of the normal equations, we find the values of the unknowns. Thus

$$\begin{aligned}
4z_1 &= (l_{12} - l_{01}) + (l_{13} - l_{02}) + (l_{14} - l_{03}) \\
4z_2 &= -(l_{12} - l_{01}) + (l_{23} - l_{12}) + (l_{24} - l_{13}) \\
4z_3 &= -(l_{23} - l_{12}) + (l_{34} - l_{23}) - (l_{13} - l_{02}) \\
4z_4 &= -(l_{34} - l_{23}) - (l_{24} - l_{13}) - (l_{14} - l_{03})
\end{aligned}$$

The computation of the unknowns may be much facilitated by arranging in tabular form, as follows:

z_1	z_2	z_3	z_4	
	$-(l_{12} - l_{01})$	$-(l_{13} - l_{02})$	$-(l_{14} - l_{03})$	Sum_1
		$-(l_{23} - l_{12})$	$-(l_{24} - l_{13})$	Sum_2
			$-(l_{34} - l_{23})$	Sum_3
$-\text{Sum}_1$	$-\text{Sum}_2$	$-\text{Sum}_3$		
$4z_1$ z_1	$4z_2$ z_2	$4z_3$ z_3	$4z_4$ z_4	

The corrections x_1, x_2, x_3 are then found from the relations

$$\begin{aligned}
x_1 &= z_1 \\
x_2 &= x_1 + z_2 \\
x_3 &= x_2 + z_3
\end{aligned}$$

since the correction x_0 may be assumed to be zero, and then x_4 must also be zero from the relations (1) and (4).

As a check we notice that the values of z_4 and x_3 must be equal but of opposite sign.

The Precision.—If v_1', v_2', . . . denote the residuals of equations 2, we have for the m. s. e. of a column length

$$\sqrt{\frac{[v'v']}{6-3}};$$

the number of equations being 6 and of independent unknowns 3.

Now, since the column length involves the difference of the calibration corrections at its ends, if we assume the m. s. e. of these corrections to be equal we shall have the m. s. e. of a calibration correction by dividing the above result by $\sqrt{2}$; thus

$$\mu = \sqrt{\frac{[v'v']}{2(6-3)}}.$$

The weights of the unknowns or of any functions of them may be found by the methods of Chapter IV.

Ex. The following were the observed values of the lengths of the 45°, 90°, and 135° columns of thermometer Green 4470, made to determine the calibration corrections at the 77°, 122°, and 167° points :

45° col.	90° col.	135° col.
44°.68	90°.07	134°.61
44°.71	90°.09	134°.68
44°.72	90°.11	
44°.75		

The corrections at the freezing and boiling points are known.

Solution. Arranging in tabular form,

z_1	z_2	z_3	z_4	Sums.
	$-0°.03$	$-0°.02$	$-0°.07$	$-0°.12$
		$-0°.01$	$-0°.02$	$-0°.03$
			$-0°.03$	$-0°.03$
$+0°.12$	$+0°.03$	$+0°.03$		
$+0°.12$	$0°.00$	$0°.00$	$-0°.12$	

$$\therefore\quad z_1 = +0°.03 \qquad z_3 = 0°.00$$
$$z_2 = 0°.00 \qquad z_4 = -0°.03$$

and

$$x_1 = z_1 = 0°.03$$
$$x_2 = x_1 + z_2 = 0°.03$$
$$x_3 = x_2 + z_3 = 0°.03$$

Also

$$\mu = \sqrt{\frac{0.0004}{6}} = \pm 0°.01$$

188. The following memoirs may be consulted: Sheepshanks, *Monthly Notices Royal Astronomical Society*, vol. xi. pp. 233–248; Hansen, *Von der Bestimmung der Theilungsfehler eines gradlinigen Maassstabes*, Leipzig, 1874; Russell, *American Journal of Science*, vol. xxi. pp. 373–379; Thorpe, *Report British Association for Advancement of Science*, 1882, pp. 145–204; Brown, *Van Nostrand's Engineering Magazine*, vol. xxix. pp. 1–7; Benoit, *Mesures de dilatation et comparisons des règles métriques*, Paris, 1883.

CHAPTER X.

APPLICATION TO EMPIRICAL FORMULAS AND INTERPOLATION.

189. In all discussions hitherto we have considered the observed quantity, whether a function of one or more variables, to be a function whose form was known and which could therefore be developed in terms of that variable. In the physical sciences we meet with a different problem. From observation we have values of a function corresponding to certain known values of a variable, and are required to determine the form of the function from the observed values; in other words, to find the algebraic formula connecting the function and the variable. The method of least squares will not enable us to find the most probable form of this function. All it will do is to show how to approach more closely to its form after that form has been found approximately by other means.

For example, the observed height of the tide at a place may be considered a function of the time of day. If observations are made with a staff gauge three or four times a day, and the stage of water at a time not observed is required, we cannot interpolate between the observed values in the most probable manner till the function connecting the height and the time of day is known.

Sometimes, indeed, the observations may be so taken as to record themselves. Thus with Saxton's self-registering tide gauge, in common use in the United States, the curve representing the rise and fall of the water is traced continuously on a web of paper moving uniformly past a pencil point by means of clock-work. Plotting the time along one edge of the paper as abscissa, the stage of water at

any time required can be read at once from the sheet. The curve representing the rise and fall of the water is in this case completely known, and no computation is necessary.

190. From the nature of physical phenomena, in which observations are made only at intervals, and into which systematic error largely enters, we must admit that it is most probable that the function connecting the observed quantity and the variable is different for each observation. We should thus have a series of equations of the form

$$\begin{aligned} M_1 &= f_1(x, a, b, c, \ldots) \\ M_2 &= f_2(x, h, k, l, \ldots) \\ \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where $M_1, M_2, \ldots$ are the observed values, x the variable, and $a, b, \ldots; h, k, \ldots$ are constants.

In order to apply the method of least squares to determine the constants, the functions must first be reduced to the linear form, and the number of quantities to be determined must be made less in number, arbitrarily it may be, than the number of observations. The first point is to determine as nearly as we can the general form of the functions connecting the observed quantities and the variable.

For this no general rule can be given. From theoretic considerations we may hit upon a certain formula as plausible. It must then be found by trial how closely it fits the observed values. It is often very convenient to make a graphical representation, the variable being the abscissa, and the observed values of the function corresponding to known values of the variable the ordinates. The points plotted will in general fall so irregularly, here a group and there a vacant space, that a free-hand curve only can be drawn which will conform to the general outline of the plot. The curve will represent the general form of the function we are seeking. This method of eliminating irregularity by drawing a mean curve to represent the value

of the function corresponds to assuming a common form for the algebraic functions $f_1, f_2, \ldots$ above.

A good example is afforded by the method employed by Humphreys and Abbot in their Mississippi River work for finding the law of velocities below the surface in a plane parallel to the current.* The observations for velocity were made with floats at different depths, started from boats anchored at various distances from the shore. In the first place, all of the velocities observed from each anchorage were plotted on cross-section paper, the depths of the floats being the ordinates and the velocities the abscissas. The resulting curves showed marked irregularities of form. To eliminate errors of observation the velocities were grouped in sets corresponding to nearly equal depths of water and to nearly equal velocities of the river, and the means plotted. These curves indicated a law, though not sufficiently clearly to allow of deducing an algebraic expression for it. "It is manifest that some further combination is necessary in order to eliminate the effect of disturbing causes. Since the absolute depths differ, this can only be done by combining the velocities at proportional depths." A curve was, therefore, plotted from the mean of all the velocities at proportional instead of at absolute depths. This curve proved to be approximately a parabola with axis horizontal and vertex about 0.3 of the depth of the river below the surface. It could, therefore, be expressed by an algebraic formula.

In many cases we may find the form of the function over which the observations extend by means of a formula of interpolation. Thus if x' is an approximate value of x, then for observations in the vicinity of x we have, by Taylor's theorem,

$$\begin{aligned} f(x) &= f\{x' + (x - x')\} \\ &= f(x') + b(x - x') + c(x - x')^2 + \ldots \end{aligned}$$

where $b, c, \ldots$ are unknown constants to be determined.

* *Physics and Hydraulics of the Mississippi River*. Washington, 1876.

Having found the form of the function approximately by such means as the preceding, we can, by the method of least squares, determine the most probable values of the constants involved, so far as the observations on hand are concerned. For let the function be reduced to the linear form, and the most probable values of the constants and the precisions of these values may then be found by the ordinary rules for observation equations as laid down in Chapter IV.

191. The question naturally arises as to the number of terms of a series, such as the above, that should be taken in any special case. Cut and try is our main guide. If the plot of the observations shows, for example, that the phenomenon can be very closely represented by a straight line, we should take the first two terms. Thus it has been usual to take the relation between the length, V, of a bar of metal and the temperature, t, to be of the form

$$V = M + b(t - t')$$

where M is the observed length at the temperature t', and b is the coefficient of expansion to be found from the observations. More refined methods of observation indicate, however, that this expression is not sufficient to express the relation between length and temperature. The plot of the observed values indicates a curve of higher dimensions than the first, so that we must take additional terms of the series to represent it, thus:

$$V = M + b(t - t') + c(t - t')^2 + \ldots$$

A second guide is afforded by the value obtained of the sum of the squares of the residuals of the observation equations. If the nature of the observations is such as to warrant the expectation of a value much smaller than that obtained, we may take additional terms of the series; and if this is still unsatisfactory we must seek another function. This test may also be used in comparing one form of function with another.

In any case, even when all of the tests applied appear satisfactory, we cannot say that our final result is the most probable value of the function that can be found. We have made too many assumptions for that. In the first place, the form of the function assumed as a first approximation is too uncertain; and, again, we have arbitrarily assumed the number of constants to be determined, and determined this limited number so as to satisfy a special set of observations only. All that our final formula will enable us to do is to interpolate in the most probable manner within the range of the special set of observed values, but not to extrapolate. If, however, the formula has been tested by many series of observations made under the most diverse conditions, and is found to satisfy them well, we can with confidence apply it in cases where no observations have been made. We may, in fact, consider that we have found the law connecting the function and the variable.

192. It is evident, too, that different experimenters may derive formulas widely different to represent like phenomena, and that each formula may satisfy the special set of observations from which it was derived tolerably well. Each experimenter may have chosen his constants differently, as well as a different form of function to begin with. As an example of this we may cite the formulas proposed * to represent the variation of the elastic forces of vapor at different temperatures t.

Young proposed

$$P = (a + bt)^m$$

a and b being constants to be found from observation.

Roche,

$$P = a\,\alpha^{\frac{x}{1 + mx}}$$

where $x = t + 20°$.

Biot and Regnault,

$$\log P = a + b\alpha^t + c\beta^t$$

* Mousson, *Physik*, vol. ii. Zurich, 1880.

Of these formulas the first involves two constants, the second three, and the third five. The first curve, therefore, requires two observations to determine it, the second three, and the third five. If a group of observations were plotted with temperatures as abscissas and pressures as ordinates, then, since the last formula represents a curve passing through five of these points, it is evident that if the observations are good this curve would in all probability pass near all intermediate points and more closely than a curve fixed by only two or three points. It should, therefore, be chosen.

Ex. 1. The following are the results of the observations made for velocity of current of the Mississippi River by Humphreys and Abbot at Carrollton and Baton Rouge in 1851. Each is the mean of 222 observations and is given at proportional depths, the whole depth being represented by unity.

Propor. depth of float below surface.	Obs. velocity in feet per second.
0.0	3.1950
0.1	3.2299
0.2	3.2532
0.3	3.2611
0.4	3.2516
0.5	3.2282
0.6	3.1807
0.7	3.1266
0.8	3.0594
0.9	2.9759

In Art. 191 it has been shown that the curve of velocities is approximately parabolic in form. As a first approximation, therefore, we take three terms of the expansion from Taylor's theorem,

$$V = a + bD + cD^2$$

where V is the velocity, D the proportional depth, and a, b, c constants to be determined from the observations.

Hence the observation equations

$$
\begin{aligned}
3.1950 &= a \\
3.2299 &= a + 0.1b + 0.01c \\
3.2532 &= a + .2b + .04c \\
3.2611 &= a + .3b + .09c \\
3.2516 &= a + .4b + .16c \\
3.2282 &= a + .5b + .25c \\
3.1807 &= a + .6b + .36c \\
3.1266 &= a + .7b + .49c \\
3.0594 &= a + .8b + .64c \\
2.9759 &= a + 0.9b + 0.81c
\end{aligned}
$$

To solve these equations proceed as in Art. 104. Take the mean and subtract from each, and we have

$$
\begin{aligned}
+0.0188 &= -0.45b - 0.285c \\
+0.0537 &= -0.35b - 0.275c \\
+0.0770 &= -0.25b - 0.245c \\
+0.0849 &= -0.15b - 0.195c \\
+0.0754 &= -0.05b - 0.125c \\
+0.0520 &= +0.05b - 0.035c \\
+0.0045 &= +0.15b + 0.075c \\
-0.0496 &= +0.25b + 0.205c \\
-0.1168 &= +0.35b + 0.355c \\
-0.2003 &= +0.45b + 0.525c
\end{aligned}
$$

Hence the normal equations

$$
\begin{aligned}
-0.2031 &= 0.8250b + 0.7425c. \\
-0.2232 &= 0.7425b + 0.7410c
\end{aligned}
$$

and $\qquad b = 0.4424 \qquad c = -0.7652 \qquad a = 3.1952$

$$\therefore \quad V = 3.1952 + 0.4424D - 0.7652D^2$$

The residuals v are found by substituting for a, b, c in the observation equations, and give

$$[vv] = 0.57$$

If we take four terms of the expansion, so that

$$V = a + bD + cD^2 + dD^3$$

and proceed similarly, we shall find

$$a = 3.1935, \qquad b = +0.4735 \qquad c = -0.8563, \qquad d = +0.0675$$

so that

$$V = 3.1935 + 0.4735\, D - 0.8563\, D^2 + 0.0675\, D^3$$

and the sum of the squares of the residuals v is given by

$$[vv] = 0.45$$

The sum of the squares of the residual errors being less than in the former case, we conclude that the observations are better represented by the formula last obtained.

It is to be borne in mind that in the application of the method of least squares only one set of measured quantities, if more than one occur in the problem, can be considered variable. The error is thrown into this set. Thus in the problem just solved the depths are supposed to be correctly measured and the error alone to occur in the measured velocities. So in finding the expansion of a body it is usual to consider all of the error as occurring in the observed lengths, and to take the thermometer readings to be correct. The justice of these assumptions may, however, very fairly be questioned. A discussion of the general problem covering such cases will be found in Art. 106.

Ex. 2. The tension, H, of saturated steam at temperature $t°$ C. is found from the formula *

$$H = a\ 10^{-\frac{bt + ct^2 + \ldots}{1 + \alpha t}} \tag{1}$$

where $\alpha = 0.0036678$, and $a, b, c, \ldots$ are constants, $n + 1$ in number, to be determined from observed values of t and H. We have also given that when

$$t = 100°\ \text{C.}, \quad H = 760^{\text{mm.}}$$

* *Travaux et Mémoires du Bureau International des Poids et Mesures*, Paris, 1881.

To find the constants $a, b, c, \ldots$ we proceed as follows:

Substituting the values of t and H in equation 1, and eliminating a, we have

$$H = 760 \; 10^{-\left(\frac{100}{1+100a} - \frac{t}{1+at}\right)b - \left(\frac{100^2}{1+100a} - \frac{t^2}{1+at}\right)c - \ldots}$$

$$= 760 \; 10^{-(bp_1 + cp_2 + \ldots)} \text{ suppose.}$$

or

$$\log 760 - \log H = bp_1 + cp_2 + \ldots \qquad (2)$$

Next compute approximate values of $b, c, \ldots$ by selecting n of the observations and substituting the observed values of t and H in Eq. 2. Let $b', c', \ldots$ denote the values found from the solution of the resulting equations, and let H' denote the corresponding value of H, so that

$$H' = 760 \; 10^{-(b'p_1 + c'p_2 + \ldots)} \qquad (3)$$

Also, let $x, y, \ldots$ denote the corrections to be added to the approximate values $b', c', \ldots$ to find the most probable values, so that

$$b = b' + x$$
$$c = c' + y$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

then, from (2),

$$\log 760 - \log \{H' + (H - H')\} = (b' + x)p_1 + (c' + y)p_2 + \ldots$$

or expanding by Taylor's theorem and remembering that, from (3),

$$\log 760 - \log H' = b'p_1 + c'p_2 + \ldots$$

we have

$$H - H' = \frac{H'}{\text{mod.}}(p_1 x + p_2 y + \ldots)$$

the linear form required. The solution may be finished in the usual way, as given in Chapter IV.

193. The preceding examples show that as soon as it has been decided what formula will apply to a special set of observations the main difficulty is in reducing this formula to the linear form. When this has been done so that the formula is in the form of an ordinary observation equation, the reduction by the method of least squares is straightforward.

In cases where the data are insufficient or contradictory it is useless to waste time on long computations, as the graphical plot will show at a glance all that the observations can show. With cross-section paper plots can be made rapidly, and by transferring to tracing linen one plot can be placed over another, so that comparisons can be readily effected and a mean value struck.

In the problem of the law of velocities in rivers, after having decided from the various plots that the curve of velocities is approximately parabolic in form, it is better to employ the method of least squares to determine the constants of the curve rather than to trust to the graphical method throughout, but only for the reason that the data available are very complete.

194. *Periodic Phenomena.*—A large class of physical phenomena is more or less periodic in character. The daily temperature throughout the year at a given place, the error of graduation of the limb of a theodolite, the magnetic declination with reference to the time, etc., are examples. The phenomenon may not be strictly periodic in that like periods succeed each other in their proper order, or that even any one period is perfect throughout. If a plot of the observed values of the function corresponding to certain values of the variable involved be made, and it indicates the periodic character of the function, we may assume as the form of the function a number of terms of the series furnished by Fourier's theorem. Each observation will give an observation equation, and from the observation equations the values of the constants in the formula will be determined. As in the cases already discussed, the final formula is to be looked on as holding only within the limits of the observations, serving as a guide for the future study of the phenomenon in question, and only to be used with great caution outside the limits of the observed values.

Suppose, then, that n observations give the values $M_1, M_2, \ldots M_n$ corresponding to the values $\varphi, \varphi + \theta, \ldots \varphi + (n - 1)\theta$ of the variable φ uniformly distributed over

the period represented by 360°; that is, such that the interval θ is $\frac{360°}{n}$.

Now, if $f(\varphi)$ is any arbitrary periodic function of a variable φ, we may, according to Fourier's theorem, write

$$f(\varphi) = X + h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) + \ldots$$

where X, h_1, h_2, ... α_1, α_2, ... are constants to be determined from the observed values of $f(\varphi)$ corresponding to assigned values of the variable φ. Substitute M_1, M_2, M_3, ... M_n for $f(\varphi)$, and φ, $\varphi + \theta$, $\varphi + 2\theta$, ... $\varphi + (n-1)\theta$ for φ, in the above equation, and we shall have n equations from which to determine a number of constants not exceeding $n - 1$.

To lighten the numerical work let us assume the arithmetic mean of the observed values as an approximate value of X, and let x denote the correction to this value, so that

$$X = \frac{[M]}{n} + x$$

Then if, as usual, is placed

$$M_1 - \frac{[M]}{n} = l_1, \quad M_2 - \frac{[M]}{n} = l_2, \ldots$$

the n observation equations may be expressed by the general formula

$$x + h_1 \sin(\alpha_1 + \varphi + m\theta) \\ + h_2 \sin(\alpha_2 + 2\varphi + 2m\theta) + \ldots - l_{m+1} = v_{m+1}$$

where m assumes all values from 0 to $n - 1$

Writing them at full length,

$$x + y_1 \cos 0 + z_1 \sin 0 + y_2 \cos 2\,0 + z_2 \sin 2\,0 + \ldots - l_1 = v_1$$
$$x + y_1 \cos \theta + z_1 \sin \theta + y_2 \cos 2\theta + z_2 \sin 2\theta + \ldots - l_2 = v_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$v_1, v_2, \ldots$ being, as usual, the errors of observation, and

$$y_1 = h_1 \sin(\alpha_1 + \varphi), \qquad y_2 = h_2 \sin(\alpha_2 + 2\varphi), \qquad \ldots$$
$$z_1 = h_1 \cos(\alpha_1 + \varphi), \qquad z_2 = h_2 \cos(\alpha_2 + 2\varphi), \qquad \ldots$$

The normal equations reduce to the simple forms (see Art. 9)

$$\begin{aligned} n x &= [l] = 0 \\ \frac{n}{2} y_1 &= \Sigma\, l_{m+1} \cos m\theta \\ \frac{n}{2} z_1 &= \Sigma\, l_{m+1} \sin m\theta \\ \frac{n}{2} y_2 &= \Sigma\, l_{m+1} \cos 2m\theta \\ \frac{n}{2} z_2 &= \Sigma\, l_{m+1} \sin 2m\theta \\ &\ldots\ldots\ldots \end{aligned}$$

where m has all values from 0 to $n - 1$.

Hence $h_1, h_2, \ldots \alpha_1, \alpha_2, \ldots$ are known, and their values, being substituted in $f(\varphi)$, give the function required.

If the initial value of $f(\varphi)$ corresponds to $\varphi = 0$, that is, if the observed values $M_1, M_2, M_3, \ldots M_n$ of $f(\varphi)$ correspond to $0, \theta, 2\theta, \ldots (n-1)\theta$, where $n\theta = 360°$, it is simpler to write the equation for $f(\varphi)$ first of all in the form

$$f(\varphi) = X + y_1 \cos\varphi + z_2 \cos 2\varphi + \ldots$$
$$+ z_1 \sin\varphi + z_2 \sin 2\varphi + \ldots$$

where

$$y_1 = h_1 \sin\alpha_1 \qquad y_2 = h_2 \sin\alpha_2, \qquad \ldots$$
$$z_1 = h_1 \cos\alpha_1 \qquad z_2 = h_2 \cos\alpha_2, \qquad \ldots$$

Then with the value $\frac{[M]}{n}$ as the approximate value of X, we

have the n observation equations and the normal equations of the same forms as before.

Hence the function is known.

195. Two cases of frequent occurrence are:

(a) $n = 10, \qquad \theta = \frac{360^\circ}{10} = 36^\circ$

The normal equations may be written

$$5y_1 = l_1 - l_6 + \{(l_2 - l_7) - (l_5 - l_{10})\}\cos 36^\circ + \{(l_3 - l_8) - (l_4 - l_9)\}\cos 72^\circ$$
$$5z_1 = \{(l_2 - l_7) + (l_5 - l_{10})\}\sin 36^\circ + \{(l_3 - l_8) + (l_4 - l_9)\}\sin 72^\circ$$
$$5y_2 = l_1 + l_6 + \{(l_2 + l_7) + (l_5 + l_{10})\}\cos 72^\circ - \{(l_3 + l_8) + (l_4 + l_9)\}\cos 36^\circ$$
$$5z_2 = \{(l_2 + l_7) - (l_5 + l_{10})\}\sin 72^\circ + \{(l_3 + l_8) - (l_4 + l_9)\}\sin 36^\circ$$

. . . .

It will be noticed that the difference of the subscripts of the l's in each parenthesis is always five, the same as the coefficient of the unknown.

(b) $n = 12, \qquad \theta = 30^\circ$

The normal equations

$$6y_1 = (l_1 - l_7) + \{(l_3 - l_9) - (l_5 - l_{11})\}\sin 30^\circ + \{(l_2 - l_8) - (l_6 - l_{12})\}\cos 30^\circ$$
$$6z_1 = (l_4 - l_{10}) + \{(l_3 - l_9) + (l_5 - l_{11})\}\cos 30^\circ + \{(l_2 - l_8) + (l_6 - l_{12})\}\sin 30^\circ$$
$$6y_2 = (l_1 + l_7) - (l_4 + l_{10}) + \{(l_2 + l_8) + (l_6 + l_{12}) - (l_3 + l_9) - (l_5 + l_{11})\}\sin 30^\circ$$
$$6z_2 = \{(l_2 + l_8) - (l_6 + l_{12}) + (l_3 + l_9) - (l_5 + l_{11})\}\cos 30^\circ$$

. . . .

The difference of the subscripts of the l's in each parenthesis is six in this case.

196. *The Precision.*—The m. s. e. of the unit of weight is found from the usual formula

$$\mu = \sqrt{\frac{[vv]}{n - n_i}}$$

where n_i is the number of constants determined.

Check of $[vv]$. Generally (Art. 100)

$$[vv] = [ll] - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \frac{[cl.2]^2}{[cc.2]} - \ldots$$

Now substitute for $[al]$, $[bl.1]$, $[cl.2]$, . . . their values from the normal equations above, and remembering that these equations contain only one unknown each,

$$[vv] = [ll] - \frac{0}{n} - \frac{(\frac{1}{2}ny_1)^2}{\frac{1}{2}n} - \frac{(\frac{1}{2}nz_1)^2}{\frac{1}{2}n} - \ldots$$

$$= [ll] - \frac{n}{2}[hh].$$

Ex. 1. The mean monthly heights of the water in Lake Michigan at Chicago below the mean level of the lake from 1860 to 1875, for the 12 months of the year 1868, were as in column M of the following table:

	M	l		M	l
	ft.	*ft.*		*ft.*	*ft.*
Jan.	1.17	+ 0.45	July.	0.11	− 0.61
Feb.	1.25	+ 0.53	Aug.	0.43	− 0.29
March.	0.59	− 0.13	Sept.	0.68	− 0.04
April.	0.68	− 0.04	Oct.	0.94	+ 0.22
May.	0.29	− 0.43	Nov.	1.05	+ 0.33
June.	0.17	− 0.55	Dec.	1.32	+ 0.60
			Mean,	0.72	

Required a formula from which the mean daily height may be found.

The period is one year, and if we assume that its 12 months correspond to 360° and that the months are of equal length, each interval Θ would be 30°.

The values of the coefficients y_1, z_1, . . . can be at once written down from (b). They are

$$y_1 = +0.517 \qquad y_2 = -0.010$$
$$z_1 = -0.194 \qquad z_2 = +0.017$$

and therefore

$$\alpha_1 = 110°\ 34' \qquad h_1 = +0.552$$
$$\alpha_2 = 149°\ 32' \qquad h_2 = -0.020$$

$$f(\varphi) = 0.72 + 0.552 \sin (110°\ 34' + \varphi) + 0.020 \sin (149°\ 32' + 2\varphi) + \ldots$$

Ex. 2. In a micrometer microscope, to find the correction for periodic error of the micrometer screw to the readings of the graduated micrometer head.

The necessary observations are made by measuring in succession on a graduated scale the distance corresponding to intervals which are aliquot parts of a complete revolution of the screw.

Let A = the value of the space on the scale.

φ = the division on the micrometer head read on at the initial reading.

The correction to the first reading will be

$$h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) + \ldots$$

and the correction to the second reading

$$h_1 \sin(\alpha_1 + A + \varphi) + h_2 \sin(\alpha_2 + 2A + 2\varphi) + \ldots$$

Hence, since the correction to the observed value M of the scale distance is the difference of these corrections, we have

$$A = M + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi) \sin\frac{A}{2} + 2h_2 \cos(\alpha_2 + A + 2\varphi) \sin A + \ldots$$

Suppose now that the scale has been read on by n different parts of the screw, so that to the observed values $M_1, M_2, \ldots M_n$ of the graduated distance correspond the values of φ,

$$\varphi, \varphi + \Theta, \ldots \varphi + (n - 1)\Theta$$

the amount by which the screw is shifted each time being $\Theta = \frac{360°}{n}$.

We should then have n equations of the form

$$A = M + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi + m\Theta) \sin\frac{A}{2}$$
$$+ 2h_2 \cos(\alpha_2 + A + 2\varphi + 2m\Theta) \sin A + \ldots$$

where m assumes all values from 0 to $n - 1$.

If we take $\frac{[M]}{n}$ as an approximate value of A, and put

$$M - \frac{[M]}{n} = l$$

the n observations may be expressed by the general formula

$$x + 2h_1 \cos(\alpha_1 + \frac{A}{2} + \varphi + m\Theta) \sin\frac{A}{2}$$
$$+ 2h_2 \cos(\alpha_2 + A + 2\varphi + 2m\Theta) \sin A + \ldots = l_m$$

where m assumes all values from 0 to $n - 1$

Expanding the cosines, the solution may be completed as in Art. 194.

The screw of the filar micrometer of the 26-in. refractor of the U. S. Naval Observatory was examined for periodic error by Prof. Hall in 1880 (*Washington Observations*, 1877).

The micrometer was placed under the Harkness dividing engine, and the distance corresponding to each $\frac{1}{10}$ of a revolution of the screw was measured by means of the micrometer belonging to the engine. The following are the means for each $\frac{1}{10}$ of a revolution:

Microm.	M	l
0.0 to 0.1	$\overset{d}{0.6221}$	+ 9
.1 " .2	.6229	+ 17
.2 " .3	.6182	− 30
.3 " .4	.6212	0
.4 " .5	.6201	− 11
.5 " .6	.6227	+ 15
.6 " .7	.6226	+ 14
.7 " .8	.6150	− 62
.8 " .9	.6189	− 23
0.9 " 1.0	0.6285	+ 73
Mean,	0.6212 = 0″.995	

Assuming these residuals to have a periodic form, required the correction to the reading of the head of the micrometer.

The observation equations are

$$
\begin{aligned}
x + y_1 \cos 0 \quad + z_1 \sin 0 \quad + y_2 \cos 2\ 0 + z_2 \sin 2\ 0 &= 9 \\
x + y_1 \cos 36^\circ + z_1 \sin 36^\circ + y_2 \cos 72^\circ + z_2 \sin 72^\circ &= 17 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \\
x + y_1 \cos 324^\circ + z_1 \sin 324^\circ + y_2 \cos 648^\circ + z_2 \sin 648^\circ &= 73
\end{aligned}
$$

The values of $y_1, z_1, \ldots$ are at once found from (a).

The final result is

$$f(\varphi) = +0''.0002 \sin\varphi + 0''.0022 \cos\varphi - 0''.0022 \sin 2\varphi + 0''.0047 \cos 2\varphi$$

Ex. 3. To find the correction for periodic error of graduation to the value of an angle measured with a theodolite in which the circle is read by two opposite microscopes, 1 and 2.

Let A = the value of the angle.

φ = the reading of the circle with microscope 1 when the telescope is pointed at the first signal.

The correction to the reading of microscope 1 for periodic error will be

$$h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) + \ldots$$

and to the reading of microscope 2, writing $180 + \varphi$ for φ,

$$- h_1 \sin(\alpha_1 + \varphi) + h_2 \sin(\alpha_2 + 2\varphi) - \ldots$$

The correction to the mean reading on the first signal is, therefore,

$$h_2 \sin(\alpha_2 + 2\varphi) + h_4 \sin(\alpha_4 + 4\varphi) + \ldots$$

and the correction to the mean reading on the second signal,

$$h_2 \sin\{\alpha_2 + 2(A + \varphi)\} + h_4 \sin\{\alpha_4 + 4(A + \varphi)\} + \ldots$$

Hence, since the correction to the observed value M_1 of the angle is the difference of these corrections, we have

$$A = M_1 + 2h_2 \cos(\alpha_2 + A + 2\varphi) \sin A + 2h_4 \cos(\alpha_4 + 2A + 4\varphi) \sin 2A + \ldots$$

Suppose now, sighting at the same two signals, that readings have been made at n different parts of the circle, so that to the observed values of the angle $M_1, M_2, \ldots M_n$ correspond the values of φ; $\varphi, \varphi + \Theta, \ldots \varphi + (n-1)\Theta$, the amount by which the circle is shifted each time being Θ. Complete the solution as in Ex. 2.

. Given the measures of the angle Falkirk–Gasport–Pekin made with a Troughton and Simms 12-in. theodolite:

	M	l
	97° 22′ 36″.80	− 1″.03
	33″.54	+ 2″.23
	31″.75	+ 4″.02
	34″.06	+ 1″.71
	38″.67	− 2″.90
	39″.63	− 4″.06
Mean,	97° 22′ 35″.77	

$$A = 97° \ 22' \ 35''.77$$
$$\varphi = 70° \ 27'$$
$$n = 6, \quad \Theta = 60°$$

The observation equations,

$$
\begin{aligned}
&y_2 \cos 0 \quad + z_2 \sin 0 \quad + y_4 \cos 0 \quad + z_4 \sin 0 \quad = + 1.03\\
&y_2 \cos 60^\circ + z_2 \sin 60^\circ + y_4 \cos 120^\circ + z_4 \sin 120^\circ = - 2.23\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&y_2 \cos 300^\circ + z_2 \sin 300^\circ + y_4 \cos 600^\circ + z_4 \sin 600^\circ = + 4.06
\end{aligned}
$$

The normal equations,

$$
\begin{aligned}
3y_2 \qquad\qquad\qquad &= \quad 4.22\\
3z_2 \qquad\qquad &= - 11.44\\
3y_4 \qquad &= - \ 1.04\\
3z_4 &= \quad 0.55
\end{aligned}
$$

Hence the correction for periodic error of any angle A measured with this instrument is given by

$$4''.10 \cos (11^\circ + A + 2\varphi) \sin A - 1''.54 \cos (91^\circ + 2A + 4\varphi) \sin 2A$$

197. Consult Bessel, *Abhandlungen*, vol. ii. p. 364; Chauvenet, *Astronomy*, vol. ii.; Brünnow, *Astronomy;* Cole, *Great Trig. Survey of India*, vol. ii.; Woodward, *Report Chief of Engineers U. S. A.*, 1876, 1879.

APPENDIX I.

HISTORICAL NOTE.

198. The first account of the method of least squares as now employed was published by Legendre in 1805, in his *Nouvelles méthodes pour la détermination des orbites des comètes.* The main points on which he insists are the generality and ease of application of the method. He states without proof the principle that the sum of the squares of the errors must be made a minimum, and deduces the arithmetic mean as a special case of this general statement. He was also the first to make use of the term least squares (*moindres quarrés*).

The first demonstration of the exponential law of error was published in 1808 by Dr. R. Adrain, of Reading, Pa., in the *Analyst* or *Mathematical Museum.** He gives two demonstrations, the second being essentially the same as that now known as Herschel's proof. After deriving the principle of minimum squares Adrain applied it to the solution of the following four problems:

(1) Supposing $a, b, c, \ldots$ to be the observed measures of any quantity x, the most probable value of x is required.

(2) Given the observed positions of a point in space, to find the most probable position of the point.

(3) To correct the dead reckoning at sea by an observation of the latitude.

(4) To correct a survey. (Ex. 5, Art. 110.)

Gauss published in 1809, in the *Theoria motus corporum cœlestium*, a third proof of the law of error, thence deducing the principle of minimum squares. He so thoroughly de-

* There is a copy of the *Analyst* containing Adrain's proofs in the library of the American Philosophical Society, Philadelphia. An account of Adrain's life will be found in the *Democratic Review* for 1844.

veloped the subject in its principles and practical applications that comparatively little has been added by later writers. An English translation of the *Theoria motus* by Admiral Davis, U. S. N., was published in 1858, and a French translation of Gauss' memoirs on least squares by Bertrand in 1855.

The main contributions to the subject, aside from those mentioned, have been made by Laplace in the fundamental principles, and by Bessel, Hansen, and Andræ in its applications to astronomical and geodetic work.

199. Many proofs of the law of error have been given. The most important are by Adrain, Gauss, Laplace, and Hagen.

That by Gauss in 1809 assumes that if a series of observations, all equally good, are made of a quantity, the arithmetic mean of the observed values is the most probable value of the quantity. The law of error is then deduced, and the principle of minimum squares follows at once.

Laplace, in his first proof (1810), deduces the principle that the sum of the squares of the errors must be made a minimum without reference to any law of error, only assuming that positive and negative errors are equally probable and that the number of observations is infinitely great. A clear account of this proof is given by Glaisher, *Mem. Roy. Astron. Soc.*, vol. xxxix., and by Meyer, *Wahrscheinlichkeitsrechnung*, pp. 440–473. See also Airy, *Theory of Errors of Observations*, second edition; Todhunter, *History of the Theory of Probability*.

Hagen's proof (1837) is founded on the hypothesis that an error of observation is the algebraic sum of an infinite number of element errors which are all of equal value and which are as likely to be positive as negative. A modified form of this proof is given in Appendix II.

Adrain's second proof is shorter than any of the three mentioned, but is not so satisfactory. This proof is also given by Sir John Herschel, *Edinburgh Review*, vol. xcii.; Airy, *Theory of Errors of Observations*, third edition; Thomson

and Tait, *Treatise on Natural Philosophy*, vol. i.; Clarke, *Geodesy*.

For more extended information on this subject the reader is referred to Merriman's very complete memoir entitled *List of Writings Relating to the Method of Least Squares*, New Haven, 1877.

APPENDIX II.

THE LAW OF ERROR.

Proof on Hagen's [Young's] Hypothesis.

200. If a quantity, T, is to be determined, and M_1 is an observed value of T, then, if the observation were perfect, we should have

$$T - M_1 = 0$$

But since, if we make a second and a third observation, we may not find the same value as we did at first, and as we can only account for the difference on the supposition that the observations are not perfect—that is, that they are affected with certain errors—we should rather write

$$\begin{aligned} T - M_1 &= \Delta_1 \\ T - M_2 &= \Delta_2 \\ T - M_3 &= \Delta_3 \end{aligned}$$

where M_1, M_2, M_3 are the observed values, and Δ_1, Δ_2, Δ_3 are the errors of the observations.

Now, an error of observation does not result from a single cause. Thus in reading an angle with a theodolite the error in the value found is the result of imperfect adjustment of the instrument, of various atmospheric changes

of want of precision in the observer's method of handling the instrument, etc. Each of these influences may be taken as the result of numerous other influences. Thus the first mentioned may include errors of collimation, of level, etc. Each of these in turn may be taken as resulting from other influences, and so on. The final influences, or element errors, as they may be called, must be assumed to be independent of one another, and each as likely to make the resultant error too large as too small—that is, as likely to be positive as negative. The number of these element errors being very great, we may, from the impossibility of assigning the limit, consider it as infinite in any case. Each element error must consequently be an infinitesimal, and for greater simplicity we may take those occurring in any one series as of the same numerical magnitude. Hence we conclude that an error of observation may be assumed to be the algebraic sum of a very great number of independent infinitesimal element errors ϵ, all equal in magnitude, but as likely to be positive as negative.

Let the number of these element errors be denoted by $2n$, as the generality of the demonstration will not be affected by supposing this infinitely great number to be even. If all of the element errors are $+$, the error $2n\epsilon$ results, and this can occur in but one way; if all but one are $+$, the error $(2n-2)\epsilon$ results, and this can occur in $2n$ ways; and generally if $n+m$ are $+$, and $n-m$ are $-$, the error $2m\epsilon$ results, and this can occur in $\dfrac{2n(2n-1)\ldots(n+m+1)}{1\,2\ldots(n-m)}$ ways.* Hence the numbers expressing the relative frequency of the errors (that is, the number of times they may be expected to occur) are equal to the coefficients in the development of the $2n^{\text{th}}$ power of any binomial.

The element errors, infinite in number, being infinitely small in comparison with the actual errors of observation, these latter may consequently be assumed to be continuous from o to $2n\epsilon$, the maximum error. If, therefore, Δ denotes

* See Todhunter's or Newcomb's *Algebra*.

the error in which $n+m$ $+\varepsilon$'s and $n-m$ $-\varepsilon$'s occur, and $\Delta+d\Delta$ denotes the consecutive error in which $n+m+1$ $+\varepsilon$'s and $n-m-1$ $-\varepsilon$'s occur, we have

$$\Delta = 2m\varepsilon$$
$$\Delta + d\Delta = (2m+2)\varepsilon$$

and therefore

$$\Delta = m d\Delta$$

Calling f the relative frequency of the error Δ, and $f+df$ that of the consecutive error $\Delta+d\Delta$, we have

$$f = \frac{2n(2n-1)\ .\ .\ .\ (n+m+1)}{\underline{n-m}!}$$

$$f+df = \frac{2n(2n-1)\ .\ .\ .\ (n+m+2)}{\underline{n-m-1}!}$$

Hence, by division,

$$\frac{f+df}{f} = \frac{n-m}{n+m+1}$$

or

$$\frac{df}{f} = -\frac{2m+1}{n+m+1}$$

$$= -\frac{2\Delta+d\Delta}{nd\Delta+\Delta+d\Delta}$$

Now, since $d\Delta$ is infinitely small in comparison with Δ, we may write

$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta+\Delta}$$

Also, since df is infinitely small in comparison with f, 2Δ is with respect to $nd\Delta+\Delta$, and we may neglect Δ in the denominator in comparison with $nd\Delta$. We have, therefore,

$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta}$$

And since Δ is infinitely small in comparison with $nd\Delta$, and $d\Delta$ is infinitely small in comparison with Δ, it follows that n must be an infinity of the second order. It is, therefore, of a magnitude comparable with $\frac{1}{(d\Delta)^2}$, and hence $n(d\Delta)^2$ must be a finite constant. Calling this constant $\frac{1}{h^2}$, we have

$$\frac{df}{f} = -2h^2\Delta\, d\Delta$$

Integrating and denoting the value of f, when $\Delta = 0$, by f_0,

$$f = f_0 e^{-h^2\Delta^2}$$

The errors being separated by the intervals $d\Delta$, so that $0, d\Delta, \ldots \Delta, \Delta + d\Delta, \ldots$ are the errors in order of magnitude, we must, in order to make the system consistent with the definition of probability, and therefore continuous, consider not so much the relative frequency of the detached errors as the relative frequency of the errors within certain limits.

Now, by the definition of probability, the probability of an error between the limits Δ and $\Delta + d\Delta$ is represented by a fraction whose numerator is the number of errors which fall between Δ and $\Delta + d\Delta$, and denominator the total number of errors committed. If we denote this probability by $\varphi(\Delta)$ we may write

$$\varphi(\Delta) = \frac{f}{\Sigma f}$$

$$= c\, e^{-h^2\Delta^2}$$

where c is a constant, Σf being necessarily a constant for the same series of observations.

201. The principle of least squares now follows readily.

Suppose that a series of observations has been made of a function T of a certain known form,

$$T = f(X, Y, \ldots)$$

in which the constants that enter are given by theory for each observation, and X, Y, . . . are the unknowns to be found.

Call M_1, M_2, . . . the observed values of T, and T_1, T_2, . . . the corresponding true values of the function, so that the errors are found from

$$T_1 - M_1 = \Delta_1$$
$$T_2 - M_2 = \Delta_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

If we knew the true values of X, Y, . . ., and therefore of T, we should have for the probabilities, before the first, second, . . . observations are made, that the errors to be expected lie between Δ_1 and $\Delta_1 + d\Delta_1$, Δ_2 and $\Delta_2 + d\Delta_2$, . . . respectively, are

$$c_1 e^{-h_1^2\Delta_1^2},\ c_2 e^{-h_2^2\Delta_2^2},\ . \ . \ .$$

where c_1, c_2, . . . h_1, h_2, . . . are constants.

The probability ψ of the simultaneous occurrence of all of these errors, which are independent of each other, is given by (Art. 5)

$$\psi = c_1 c_2, \ . \ . \ . \ e^{-[h^2\Delta^2]}$$

But as only the observed values M_1, M_2, . . . are known, the true values of T, X, Y, . . ., Δ_1, Δ_2, . . ., and therefore of ψ, are unknown.

If now arbitrary values of X, Y, . . . are assumed, T, Δ_1, Δ_2, . . . will receive values, and therefore a value of ψ will be determined. Other assumed values of X, Y, . . . will give other values of ψ, which is therefore a function of X, Y, . . . Of all possible values which are given to X, Y, . . . there must be some one set of values which is to be chosen in preference to any others. The most probable set is naturally the one that would be chosen.

Let, then, X_1, Y_1, . . . denote the most probable values of X, Y, . . . Substitute them in the function and let

$V_1, V_2, \ldots$ denote the resulting values of $T_1, T_2, \ldots$ Then we have no longer the true errors $T_1 - M_1, T_2 - M_2, \ldots$, but the errors $V_1 - M_1, V_2 - M_2, \ldots$, which may be called *residual errors* of observation, being the difference between the most probable and the observed values. They are denoted by the symbols $v_1, v_2, \ldots$

Now, assuming that $c\,e^{-h^2v^2}$ denotes the probability of a residual between v and $v + dv$, the expression for ψ becomes

$$\psi = c_1 c_2 \ldots e^{-[h^2v^2]}$$

and the most probable set of values of $X, Y, \ldots$ would be that which corresponds to the maximum value of this expression, which can only happen when

$$[h^2v^2] \text{ is a minimum}$$

for the same set of values of $X, Y, \ldots$

This is the principle of least squares.

202. The following memoirs may be consulted for other presentations of this proof: Young, *Philosophical Transactions*, London, 1819; Hagen, *Grundzüge der Wahrscheinlichkeitsrechnung*, Berlin, 1837; Wittstein, *Lehrbuch der Differential- und Integralrechnung*, Hanover, 1849; Price, *Infinitesimal Calculus*, vol. ii., Oxford, 1865; Tait, *Edinburgh Transactions*, 1865; Kummell, *Analyst*, vol. iii., Des Moines, 1876; Merriman, *Journal Franklin Institute*, Philadelphia, 1877.

TABLE I.

Values of $\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^{\rho \frac{a}{r}} e^{-t^2}\,dt.$

$\frac{a}{r}$	$\Theta(t)$	Diff.	$\frac{a}{r}$	$\Theta(t)$	Diff.
0,0	0,000	54	2,5	0,908	13
0,1	0,054	53	2,6	0,921	10
0,2	0,107	53	2,7	0,931	10
0,3	0,160	53	2,8	0,941	9
0,4	0,213	51	2,9	0,950	7
0,5	0,264	50	3,0	0,957	6
0,6	0,314	49	3,1	0,963	6
0,7	0,363	48	3,2	0,969	5
0,8	0,411	45	3,3	0,974	4
0,9	0,456	44	3,4	0,978	4
1,0	0,500	42	3,5	0,982	3
1,1	0,542	40	3,6	0,985	2
1,2	0,582	37	3,7	0,987	3
1,3	0,619	36	3,8	0,990	1
1,4	0,655	33	3,9	0,991	2
1,5	0,688	31	4,0	0,993	1
1,6	0,719	29	4,1	0,994	1
1,7	0,748	27	4,2	0,995	1
1,8	0,775	25	4,3	0,996	1
1,9	0,800	23	4,4	0,997	1
2,0	0,823	20	4,5	0,998	0
2,1	0,843	19	4,6	0,998	0
2,2	0,862	17	4,7	0,998	1
2,3	0,879	16	4,8	0,999	0
2,4	0,895	13	4,9	0,999	0
2,5	0,908		5,0	0,999	

TABLE II.

Factors for Bessel's Probable-Error Formulas.

n	$\frac{.6745}{\sqrt{n-1}}$	$\frac{.6745}{\sqrt{n(n-1)}}$	n	$\frac{.6745}{\sqrt{n-1}}$	$\frac{.6745}{\sqrt{n(n-1)}}$
			40	0.1080	0.0171
			41	.1066	.0167
2	0.6745	0.4769	42	.1053	.0163
3	.4769	.2754	43	.1041	.0159
4	.3894	.1947	44	.1029	.0155
5	0.3372	0.1508	45	0.1017	0.0152
6	.3016	.1231	46	.1005	.0148
7	.2754	.1041	47	.0994	.0145
8	.2549	.0901	48	.0984	.0142
9	.2385	.0795	49	.0974	.0139
10	0.2248	0.0711	50	0.0964	0.0136
11	.2133	.0643	51	.0954	.0134
12	.2029	.0587	52	.0944	.0131
13	.1947	.0540	53	.0935	.0128
14	.1871	.0500	54	.0926	.0126
15	0.1803	0.0465	55	0.0918	0.0124
16	.1742	.0435	56	.0909	.0122
17	.1686	.0409	57	.0901	.0119
18	.1636	.0386	58	.0893	.0117
19	.1590	.0365	59	.0886	.0115
20	0.1547	0.0346	60	0.0878	0.0113
21	.1508	.0329	61	.0871	.0111
22	.1472	.0314	62	.0864	.0110
23	.1438	.0300	63	.0857	.0108
24	.1406	.0287	64	.0850	.0106
25	0.1377	0.0275	65	0.0843	0.0105
26	.1349	.0265	66	.0837	.0103
27	.1323	.0255	67	.0830	.0101
28	.1298	.0245	68	.0824	.0100
29	.1275	.0237	69	.0818	.0098
30	0.1252	0.0229	70	0.0812	0.0097
31	.1231	.0221	71	.0806	.0096
32	.1211	.0214	72	.0800	.0094
33	.1192	.0208	73	.0795	.0093
34	.1174	.0201	74	.0789	.0092
35	0.1157	0.0196	75	0.0784	0.0091
36	.1140	.0190	80	.0759	.0085
37	.1124	.0185	85	.0736	.0080
38	.1109	.0180	90	.0713	.0075
39	.1094	.0175	100	.0678	.0068

TABLE III.

Factors for Peters' Probable-Error Formulas.

n	$\frac{.8453}{\sqrt{n(n-1)}}$	$\frac{.8453}{n\sqrt{n-1}}$	n	$\frac{.8453}{\sqrt{n(n-1)}}$	$\frac{.8453}{n\sqrt{n-1}}$
			40	0.0214	0.0034
			41	.0209	.0033
2	0.5978	0.4227	42	.0204	.0031
3	.3451	.1993	43	.0199	.0030
4	.2440	.1220	44	.0194	.0029
5	0.1890	0.0845	45	0.0190	0.0028
6	.1543	.0630	46	.0186	.0027
7	.1304	.0493	47	.0182	.0027
8	.1130	.0399	48	.0178	.0026
9	.0996	.0332	49	.0174	.0025
10	0.0891	0.0282	50	0.0171	0.0024
11	.0806	.0243	51	.0167	.0023
12	.0736	.0212	52	.0164	.0023
13	.0677	.0188	53	.0161	.0022
14	.0627	.0167	54	.0158	.0022
15	0.0583	0.0151	55	0.0155	0.0021
16	.0546	.0136	56	.0152	.0020
17	.0513	.0124	57	.0150	.0020
18	.0483	.0114	58	.0147	.0019
19	.0457	.0105	59	.0145	.0019
20	0.0434	0.0097	60	0.0142	0.0018
21	.0412	.0090	61	.0140	.0018
22	.0393	.0084	62	.0137	.0017
23	.0376	.0078	63	.0135	.0017
24	.0360	.0073	64	.0133	.0017
25	0.0345	0.0069	65	0.0131	0.0016
26	.0332	.0065	66	.0129	.0016
27	.0319	.0061	67	.0127	.0016
28	.0307	.0058	68	.0125	.0015
29	.0297	.0055	69	.0123	.0015
30	0.0287	0.0052	70	0.0122	0.0015
31	.0277	.0050	71	.0120	.0014
32	.0268	.0047	72	.0118	.0014
33	.0260	.0045	73	.0117	.0014
34	.0252	.0043	74	.0115	.0013
35	0.0245	0.0041	75	0.0113	0.0013
36	.0238	.0040	80	.0106	.0012
37	.0232	.0038	85	.0100	.0011
38	.0225	.0037	90	.0095	.0010
39	.0220	.0035	100	.0085	.0008